Tarski fixed point theorem

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Exercise 5.8 The set S is defined to be the least subset of natural numbers $\mathbb{N}$ such that:

$1 \in S$;

if $n \in S$, then $3n \in S$;

if $n \in S$ and $n > 2$, then $(n-2) \in S$.

Show that $S = \{m \in \mathbb{N} \mid \exists r, s \in \mathbb{N} \cup \{0\}.\ \ m = 3^r - 2s\}$. Deduce that S is the set of odd numbers. □

By prop 2, we only need Q is R-closed.

for all $(X/y) \in R$

$(\forall x \in X.\ x \in I_R \ \&\ P(x)) \Rightarrow (\underline{y \in I_R} \ \&\ P(y))$

Because I_R is R-closed,

$\forall x \in X.\ x \in I_R \Rightarrow y \in I_R$ follows.

then $\boxed{(\forall x \in X.\ x \in I_R \ \&\ P(x)) \Rightarrow P(y)}$

principle of rule induction

Q: I_R or induction $\xleftrightarrow{\text{relation?}}$ $\boxed{\text{Schröder-Bernstein}}$

or fixed point

R is a set of rule. $I_R = \bigcap \{Q \mid Q \text{ is } \boxed{R\text{-closed}}\}$

$\forall X \subseteq Q,\ (X/y) \in R$

$y \in Q$.

A set B of propositions
$\{x_1 \cdots x_i,\ y_1 \cdots y_k\}$

Define the action of R on B

$$\hat{R}(B) = \{ y \mid \exists X \subseteq B.\ (X/y) \in R \}$$

proposition: B is R-closed iff $\hat{R}(B) \subseteq B$.

$\Longrightarrow$ easy by definition.

$\Longleftarrow$

$\hat{R}$ is a $\boxed{\dfrac{\text{monotone}}{\text{monotonic}}}$ operation.

$$A \subseteq B \Rightarrow \hat{R}(A) \subseteq \hat{R}(B)$$

start from ϕ. $\boxed{\text{id}}$

$$A_0 = \hat{R}^0(\phi) = \phi$$

$$A_1 = \hat{R}^1(\phi) = \hat{R}(\phi)$$

$$A_2 = \hat{R}^2(\phi)$$

$\vdots$

A_1 consists all the axioms

A_{n+1} is the conclusions immediately follows from A_n.

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots A_n \subseteq A_{n+1} \subseteq \cdots$$

$$\boxed{A = \bigcup_{n \in \mathbb{N}_0} A_n}$$

Assume

Theorem: rules in R are all finitary — finite premises

$$\frac{x_i \ \cdots \ x_n}{y}$$

then ① A is R-closed. ✓

② $\hat{R}(A) = A$. ✓

③ A is the least R-closed set. $\longrightarrow$ $A =$

proof: Take B R-closed and — We want $A \subseteq B$

$\Rightarrow \hat{R}(B) = B$

Now we show $\forall n \in \mathbb{N}_0$, $A_n \subseteq B$.

Base $A_0 \subseteq B$ of course. $\emptyset \subseteq$ any set ✓

Inductive: assume $A_n \subseteq B$.

$$A_{n+1} = \hat{R}(A_n) \subseteq \hat{R}(B) = B.$$

好了

Tarski's Fixed point Theorem

Thm (minimal version) U is a set, $P(U)$ power set,

Let $\varphi : P(U) \to P(U)$ monotonic (total) function.

Define. $m = \bigcap \{ S \subseteq U \mid \varphi(S) \subseteq S \}$.

Then. m is a fixed point of φ.

Moreover, m is a least pre-fixed point of φ.

S s.t. $\varphi(S) \subseteq S$

Proof: $X = \{ S \subseteq U \mid \varphi(S) \subseteq S \}$ $\boxed{m = \bigcap X}$

Take $S \in X$, then $m \subseteq S$.

$\varphi(m) \subseteq \varphi(S)$ by monotonicity.

$\varphi(S) \subseteq S$

Then. $\varphi(m) \subseteq S$ for any $S \in X$.

$\varphi(m) \subseteq \bigcap X = m$ $\Rightarrow$ $\boxed{m \text{ is a pre-fixed point.}}$

$\varphi(m) \subseteq \bigcap X = m$ $\Rightarrow$ $\boxed{m \text{ is a pre-fixed point}}$

$\Downarrow$

$\varphi(m) \subseteq m$ $\Rightarrow$ m is the minimal pre-fix point. by def.

Now: $\varphi(m) \supseteq m$ or $m \subseteq \varphi(m)$

$\varphi(\underbrace{\varphi(m)}_{S}) \subseteq \underbrace{\varphi(m)}_{S}$

Then. $\varphi(m) \in X = \{ S \subseteq U \mid \varphi(S) \subseteq S \}$ ✓

Then. $m \subseteq \varphi(m)$