$\prec$

Set A. a well-founded relation $\prec$ is a binary relation

s.t. any descending chain $a_N \cdots \prec a_n \cdots \prec a_1 \prec a_0$ has a minimal element.

When $a \prec b$, a is a predecessor of b

non-e.g. $(\mathbb{R}, \leq)$ is not well founded.

① Emmy Noether

② Axiom of choice: If $\bigcup_{i \in I} A_i$, then $\exists\, \mathcal{A} = \{a_1 \cdots a_i\}$ s.t. $a_i \in A_i$.

$\Updownarrow$

Zorn' Lemma. $\Updownarrow$ Any set B, if every descending chain has a lower bound, then it has a minimal element.

Principle of well-founded induction:

$\prec$ is a well founded relation on set A,

Want: $\forall a \in A$, $P(a)$,

Only need: $\forall a \in A$, $[\forall b \prec a\ P(b)] \Rightarrow P(a)$

Trivial e.g. $(\mathbb{N}_0, <)$ is well founded.

well founded induction is standard induction.

$P(0)$ 0 has no predecessor.

$[n \prec n+1, P(n)] \Rightarrow P(n+1)$

Create new well fonded relations from old ones.

△ Product relation

$\boxed{\prec_1 \text{ on } A_1 \qquad \prec_2 \text{ on } A_2}$

on $A_1 \times A_2$

$$(a_1, a_2) \prec (a_1', a_2') \overset{def}{\Longleftrightarrow} a_1 \prec a_1' \text{ and } a_2 \prec a_2'$$

product relation, well founded.

△ Lexicographic product relation

on $A_1 \times A_2$

$$(a_1, a_2) \prec (a_1', a_2') \text{ iff } a_1 \prec a_1'$$

$$\text{or } (a_1 = a_1' \text{ and } a_2 \prec a_2')$$

A non-trivial example of well fonded induction:

Eulerian graphs

graph (V, E) $\boxed{\begin{array}{l} V \text{ set of vertices} \\ E \text{ set of edges} \end{array}}$ finite.

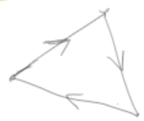
edges $\to \{v, v'\}$

a connected graph: any pair v, v' are connected by a path of edges $\{v_0, v_1\}\{v_1, v_2\} \ldots \{v_{n-1}, v_n\}$, $v_0 = v$, $v_n = v'$.

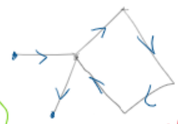
a circuit (loop) of a graph is a path of edges $\{v_0 v_1\}\{v_1 v_2\} \cdots \{v_{n-1} v_n\}$ where $v_0 = v_n$

Q (7 bridges in Königsberg) When does a connected graph have a $\boxed{\text{Eulerian}}$ circuit?

this path visits each edge exactly once.

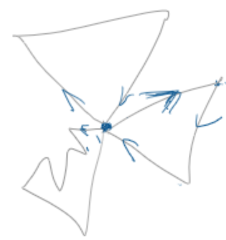


either all vertices have even degree or



not a circuit.

exactly two vertices have odd degree and all others have even degree



Thm (Euler) A finite connected graph has an Eulerian circuit iff every vertex has even degree.

$\Longrightarrow$ $\Longleftarrow$

~~If:~~

only if $\Longrightarrow$: We have an Eulerian circuit.

Follow this path: on each vertex, there is an incoming edge and an outgoing edge.

if $\Longleftarrow$: Recall: $P(S)$, $S_1 \subseteq S_2$ is well founded.

$\{\text{finite graph}\}$ $<$ well founded

product relation

$G_1 < G_2$ iff $V_1 \subseteq V_2$ and $E_1 \subseteq E_2$

Let G finite connected graph s.t. each vertex has even degree.

want an Eulerian circuit for G.

Assume $\forall G'$ s.t. $G' < G$ s.t. each vertex has even degree, G' has an Eulerian circuit.

Now build a Eulerian circuit of G.

step I: find a circuit C in the graph G.

Find a maximal path, s.t. no edge appears more than once.

such path must have a loop

because the graph is finite

path has to connect back to the starting point.

① path itself is a loop

Done

② path contains loop and something else.

C

step 2: remove the loop C.

This results in one or more connected graph G'('s)

By induction hypothesis, each G' has an Eulerian circuit.

Linking those circuit to C leads to an Eulerian circuit of G.

Define sets recursively according to a well-founded relation.

domain

set B , $<$.

$F(b, c_1 \cdots c_k \cdots\cdots)$ is an expression,

could be infinite

s.t. $\forall\ b \in B,\ c_1, \cdots c_k \cdots \in C,\ F(b, c_1 \cdots c_k \cdots) \in C$

set C.

Then a recursive definition of the form

$$f(b) = F(b,\ f(b_1),\ f(b_2) \cdots f(b_k), \cdots)$$

where $b_1 < b$, $b_2 < b$, $\cdots b_k < b \cdots\cdots$

determines a total function $f: B \to C$.

E.g. $(\underbrace{\mathbb{N}_0}_{B},\ \underbrace{<}_{<})$ $\qquad \underbrace{\mathbb{N}_0}_{C}$ $\qquad f = \underbrace{fib(n)}_{? \in \mathbb{N}}$

$fib(0) = 0 \qquad fib(1) = 1$

$$fib(n) = fib(n-1) + fib(n-2) \quad \forall n > 1.$$

Well-founded recursively defined sets are the collection of all definable/computable sets.