Computability and diagonal argument

program = $\underbrace{\{01\cdots\cdots\}}_{\text{finite.}}$

{program} → countable.

How many computational $\boxed{\text{problems}}$ do we have? ← $\boxed{\text{decision problem}}$

A: card $\left\{ \underbrace{\{01\cdots\cdots\}}_{\text{input.}} \mapsto 0 \text{ or } 1. \right\}$

= uncountable.

(Deterministic) Finite Automata.

$$FA = (Q, q_0, F, \boxed{\Sigma}, \delta)$$

alphabets

$\Sigma \to \{0,1\}$

Q : finite set of states

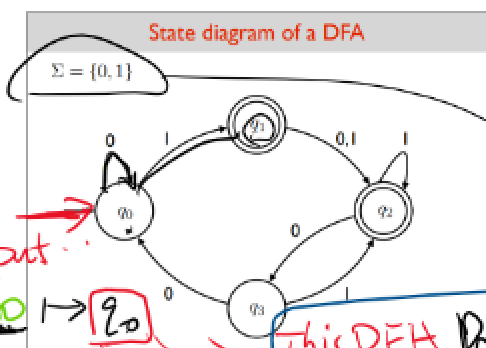
$q_0 \in Q$: intial state

$F \subseteq Q$: Set of final states

Σ : input alphabeta

δ : transition function.

Any (D)FA. has a unique directed graph representation



$Q = \{q_0, q_1, q_2, q_3\}$

$q_0 = q_0$ initial

$F = \{q_1, q_2\}$

$\Sigma = \{0,1\}$

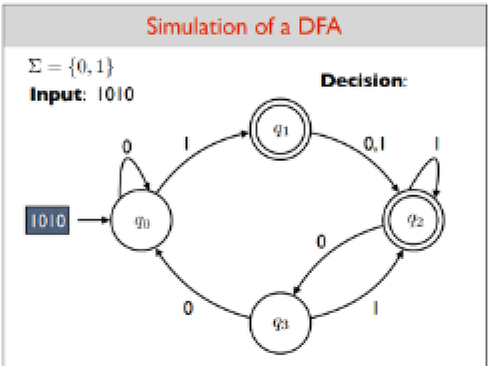
$\delta : Q \to Q$

input: $0 \mapsto q_0$

$1 \mapsto q_1$

final: This DFA

This DFA Does NOT computes accept decide the word {0} {1}!

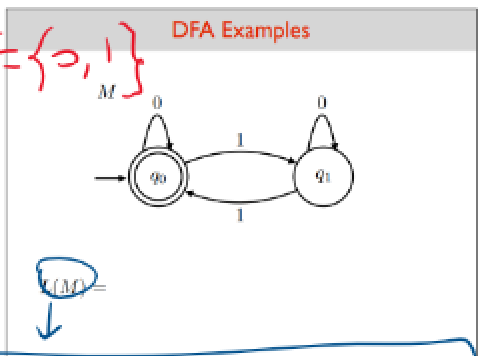


$q_0 \xrightarrow{1} q_1 \xrightarrow{0} q_2 \xrightarrow{1} q_2 \xrightarrow{0} q_3$

output is q_3

This DFA does not accept 1010.

$\Sigma = \{0, 1\}$



Set of words M accepts ?

even number of 1's

?

Def: A language is regular if it is accepted by some DFA.

Regular operation:

$A = \{01, \ldots\}$ and $B = \{01 \ldots\}$ are some language

* Union: $A \cup B$

Regular operation :

A and B are some language

$A = \{\ldots\}$, $B = \{\ldots\}$

- Union : $A \cup B$
- Concatenation. $A \circ B = \{ xy \mid x \in A \ \& \ y \in B \}$
- Kleene star : $A^* = \bigcup_{k \geq 0} A^k$

start of Church

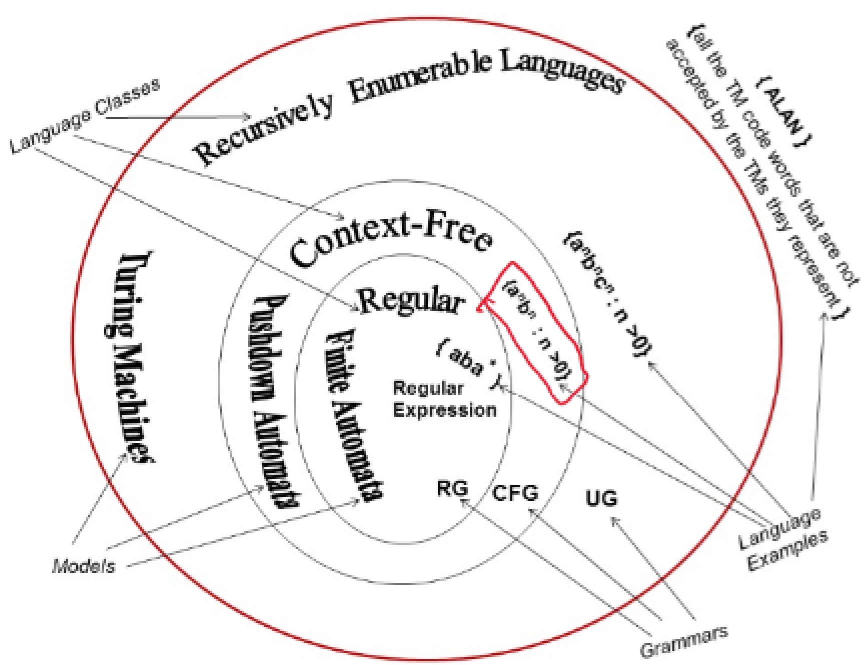
$A^0 = \{ \varepsilon \}$ empty word

$A^1 = A$

$A^2 = A \circ A$

$A^k = \underbrace{A \circ A \circ \cdots \circ A}_{k}$

$$\boxed{\text{Regular language is closed under Regular operations}}$$



$\frac{0}{k}$

$0^n 1^n$ $\forall n \in \mathbb{N}_0$

is not regular.

$\Downarrow$

not acceptable by FA.

- $q_0 \in Q$ is the initial state. ✓
- $F \subseteq Q$ is the set of distinguished states called final states.
- Σ is the input alphabet. 0, 1
- Γ is the tape alphabet, here $\Sigma \subseteq \Gamma$ and $B \in \Gamma$. → blank
- δ are the transitions,

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$$

final states → accept / reject → This TM Halts.

$\Updownarrow$

looping

$$(x+1)^2 \overset{?}{=} x^2+2x+1 \quad (0,1)$$

$$0 \overset{?}{=} 0 \Rightarrow \text{output is 1.}$$

Example 1. Let $\Sigma = \{0, 1\}$, $\Gamma = \{0, 1, X, \mathbb{B}\}$, $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$, $F = \{q_5\}$ and $\delta : Q \times \Gamma \to Q \times \Gamma \times \{\leftarrow, \rightarrow\}$ the transition function given for the following table:

	0	1	X	$\mathbb{B}$
q_0	$(q_1, X, \rightarrow)$			$(\mathbb{B}, \mathbb{B}, \rightarrow)$
q_1 $(q_1, 0, \rightarrow)$	$(q_2, X, \rightarrow)$			
q_2	$(q_3, X, \leftarrow)$			
q_3	$(q_3, 0, \leftarrow)$	$(q_3, 1, \leftarrow)$	$(q_3, X, \leftarrow)$	$(q_4, \mathbb{B}, \rightarrow)$
q_4	$(q_1, X, \rightarrow)$		$(q_4, X, \rightarrow)$	
q_5				

where the empty entries mean that there is no transition defined. The input that the machine accepts are strings that look like 0^n1^n.

not defined

input 0 $\quad (q_0, 0) \to (q_1\ X \rightarrow) \to \square$

input 1 $\quad (q_0, 1) \to \square$ not accept

input 10 $\quad \to \square$ not accept.

input 01 $\quad (q_0, 0) \to (q_1\ X \rightarrow)$

Decider:

$\Sigma = \{0, 1\}$

$\Sigma^* = \{$all possibly $w = 01\ldots\}$

Σ^*

A TM is a decider if it halts on every word $w \in \Sigma^*$.

Also, $L(M) = \{ w \in \Sigma^* \mid M \text{ accept } w \}$

If $L \subseteq \Sigma^*$ that is accepted by some TM, then L is semi-decidable.

If $L = L(M)$ for some decider M, then L is decidable.

Halting problem:

Language → $A_{TM} = \{ \langle \underbrace{M}_{\text{Turing machine}}, \underbrace{w}_{\text{some word}} \rangle \mid$ T/M M accept input w. $\}$

Q: Is A_{TM} semi-decidable?

A: A_{TM} is undecidable.

Proof:

Step 1: Construct a language D, s.t. D is undecidable

key

Step 2: Show if A_{TM} is semi-decidable, so is D.

key step: Use the diagonalization method: Fix a coding method

$\{$ all TM $\}$ is countable $\rightarrow \{ M_i \}$

$$D = \{ \underset{0,1,\dots}{\langle M_i \rangle} \mid \text{TM } M_i \text{ does not accept input } \langle M_i \rangle \}$$

claim: D is not decidable

pf: Suppose $\boxed{D = L(M_n)}$

Then: is $\langle M_n \rangle \in L(M_n)$?

If yes, $\langle M_n \rangle \in L(M_n)$

$$\Downarrow$$

$$\langle M_n \rangle \notin D$$

$$\Rightarrow \langle M_n \rangle \notin L(M_n)$$

If no, $\langle M_n \rangle \notin L(M_n)$

$$\Downarrow$$

$\langle M_n \rangle \in D$

$$\Downarrow$$

$\langle M_n \rangle \in L(M_n)$

Done with step 1.

Done with step 1.

Step 2: Assume A_{TM} is decidable by a TM H

Halting

D now can be interpreted as

"On input $\langle M_i \rangle$, simulate H on $\langle M_i, \langle M_i \rangle \rangle$,

若 H accept its input, then reject;

若 H reject its input, then accept.

$\Sigma = \{0, 1\}$

$\Sigma^* = \{$ all possible words $= 0 1 \ldots \}$

Decider:

Σ^*

A TM is a decider if it halts on every word $w \in \Sigma^*$.

Also, $L(M) = \{ w \in \Sigma^* \mid M \text{ accept } w \}$