Model theory

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primitive (Boolean) propositions $\qquad p \quad q$

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Compound statements

$\neg$	$\vee$	$\wedge$	logic
A^c	$A \cup B$	$A \cap B$	set

U ambient set

$\boxed{\{x \in U \mid P(x)\}}$ → property.

extension of $p(x)$ in U.

$U = \mathbb{R}$

$\{ y \in \mathbb{R} \mid y > 2 \}$

properties	
$p(x)$	$\{x \in U \mid p(x)\}$
$P(x)$ and $Q(x)$	$\{x \in U \mid p(x)\} \cap \{x \in U \mid q(x)\}$
or	$\cup$
$\neg P(x)$	$\{x \in U \mid p(x)\}^c$ → comp in U
$Q(x) \Rightarrow P(x)$	$\{x \in U \mid Q(x)\}^c \cup \{x \in U \mid p(x)\}$
$\neg Q(x) \vee P(x)$	

✓ First order logic $\qquad p(x) \quad (\neg, \wedge, \vee)$

quantifier on x.

Second order logic = FOL + quantifiers on predicates.

A foundation of math

Model.

M. $\qquad$ universe of states

U_M

For any proposition A, we associate an interpretation

$$[\![A]\!]_M \subseteq U_M,$$

s.t.

$$[\![T]\!]_M = U_M$$

$$[\![F]\!]_M = \emptyset$$

$$[\![A \wedge B]\!]_M = [\![A]\!]_M \cap [\![B]\!]_M$$

$$[\![A \vee B]\!]_M = [\![A]\!]_M \cup [\![B]\!]_M$$

$$[\![A \wedge B]\!]_M = [\![A]\!]_M \cap [\![B]\!]_M$$

$$[\![A \vee B]\!]_M = [\![A]\!]_M \cup [\![B]\!]_M.$$

$$[\![\neg A]\!]_M = [\![A]\!]_M^c$$

S model $\qquad U_S \to$ students

$$[\![a]\!]_S = \{x \in U_S \mid x \text{ is math major}\}$$

$$[\![b]\!]_S = \{x \in U_S \mid x \text{ has a driver licence}\}$$

$$[\![c]\!]_S = \{x \in U_S \mid x \text{ is nineteen}\}$$

Def: Proposition A, $\quad$ A is valid in M

iff $[\![A]\!]_M = U_M$

prop A, B $\quad$ A entails B in M

iff $[\![A]\!]_M \subseteq [\![B]\!]_M$

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prop A is valid iff it's valid in all models

$\models A$

Prop A, B, $\quad$ A entails B iff A entails B in all models

$\wedge \quad \vdash \quad \supset$

$\downarrow$ $\quad A \models B$ $\quad$ in all mod[els]

Prop A, B $\quad$ A and B are equivalent iff $A \models B$ and $B \models A$.

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proposition 2.11

Prop A, B.

$A \models B$ iff $\models (A \Rightarrow B)$

proof:

In any model M, $[\![A]\!]_M \subseteq [\![B]\!]_M$

$$\Updownarrow$$

$$[\![A]\!]_M^c \cup [\![B]\!]_M = U_M$$

$$\Updownarrow$$

$$[\![\neg A \vee B]\!]_M = U_M$$

$$\Updownarrow$$

$$\models (A \Rightarrow B)$$

A special model : Truth assignment.

1) Special model: Truth assignment.