Relations and Functions

Recall:

ordered pair (a, b), $a \in A$ and $b \in B$

Remark: ordered. $\underset{\text{if } a \neq b}{(a,b) \neq (b,a)}$ $A = B$

Cartesian product of A and B: $A \times B = \{ (a,b) \mid a \in A \text{ and } b \in B \}$.

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n} \longrightarrow \underbrace{\mathbb{R}^\infty}_{\text{Hilbert space}}$$

Remark: $\{ \{a\}, \{a,b\} \}$ behaves like (a,b)

i.e. $\{ \{a\}, \{a,b\} \} = \{ \{a'\}, \{a', b'\} \}$ iff $a = a'$ and $b = b'$

Proof: when $a \neq b$ straightforward.

when $a = b$ degeneration

Relation: $R \subseteq X \times Y$, a relation R between X and Y.

xRy for $(x,y) \in R$.

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partial function: $f \subseteq X \times Y$ is a relation, s.t.

$\forall x, y$ and y' $(x,y) \in f$ and $(x,y') \in f$ $\Rightarrow y = y'$

(total) function: is a partial function that $\forall x \in X$ $\exists y$ s.t. $(x,y) \in f$.

X domain Y codomain

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prop: ① $R, R' \subseteq X \times Y$, then $R = R'$ iff $\forall x \in X$ $\forall y \in Y$, $xRy \Leftrightarrow xR'y$.

② partial $f, f' \subseteq X \times Y$, then $f = f'$ iff $\forall x \in X$ s.t. $(x,y) \in f$, $f(x) = f'(x)$

③ total $f, f' \subseteq X \times Y$ $f = f'$ iff $\forall x \in X$ $f(x) = f'(x)$.

Composing Functions and Relations

codomain

prop: ① $R, R' \subseteq X \times Y$, then $R = R'$ iff
$\forall x \in X\ \forall y \in Y,\ xRy \Leftrightarrow xR'y$.

② partial $f, f' \subseteq X \times Y$, then $f = f'$ iff
$\forall x \in X$ s.t. $(x, y) \in f$, $f(x) = f'(x)$

③ total $f, f' \subseteq X \times Y$
$f = f'$ iff $\forall x \in X$. $f(x) = f'(x)$.

Composing Functions and Relations

$$\begin{tikzcd} X \arrow[r, squiggly, "R"] \arrow[rr, bend right, "S \circ R"'] & Y \arrow[r, squiggly, "S"] & Z \end{tikzcd}$$

$$S \circ R := \{ (x, z) \in X \times Z \mid \exists y \in Y, \text{ s.t. } (x, y) \in R \text{ and } (y, z) \in S \}$$

Associativity: $T \circ (S \circ R) = (T \circ S) \circ R$

$$\begin{tikzcd} A \arrow[r, "f"] & X \arrow[r, "f"] \arrow[rr, bend right] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

$g \circ f(x) = g(f(x))$

$\forall X$ set, $\exists$ identity relation id_X

$$id_X := \{ (x, x) \mid x \in X \}$$

Now for functions:

claim 1: $X \xrightarrow{f} Y \xrightarrow{g} Z$, f, g partial functions $\Rightarrow g \circ f$ is partial

claim 2: $X \xrightarrow{f} Y \xrightarrow{g} Z$ f, g total $\Rightarrow g \circ f$ is total

\ function $f: X \to Y$ is surjective (or onto)

if $\forall y \in Y\ \exists x \in X$ s.t. $(x,y) \in f$
$y = f(x)$

– – – – is injective (or 1-1)

if $\forall x, x' \in X$, $f(x) = f(x') \Rightarrow x = x'$.

– – – – is bijective (bijective correspondence / 1-1 correspondence)

if f is both injective and surjective.

A function $X \xrightarrow{f} Y$ has an inverse $Y \xrightarrow{g} X$

if $g(f(x)) = x$ $\forall x \in X$ and $f(g(y)) = y$ $\forall y \in Y$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "\mathrm{id}_X"'] & Y \arrow[r, "g"] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "g"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

$g \circ f$ $\qquad$ $f \circ g$

Thm: f is bijective iff it has an inverse.

$\downarrow$

$\boxed{\text{?g.48}}$

Compositions of inj / surj / bij / maps

$_1 \circ f_2 \circ \cdots \circ f_n$ are inj / surj / bij.

Direct and inverse image under a relation

$R \subseteq X \times Y$ $\qquad$ R acts on $A \subseteq X$, $B \subseteq Y$.

t image.

$RA := \{ y \in Y \mid \exists x \in A,\ (x,y) \in R \}$

erse image

$R^{-1}B := \{ x \in X \mid \exists y \in B,\ (x,y) \in R \}$

For functions we can define $f(A)$ and $f^{-1}(B)$

Compositions of inj / surj / bij / maps

$f_1 \circ f_2 \circ \cdots \circ f_n$ are inj / surj / bij.

Direct and inverse image under a relation

$R \subseteq X \times Y$ R acts on $A \subseteq X$, $B \subseteq Y$.

Direct image.

$$RA := \{ y \in Y \mid \exists x \in A.\ (x,y) \in R \}$$

Inverse image

$$R^{-1}B := \{ x \in X \mid \exists y \in B.\ (x,y) \in R \}$$

For functions we can define $\underline{f(A)}$ and $\underline{f^{-1}(B)}$

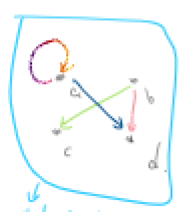
Rmk: direct & inverse images preserve set operations:

$$f^{-1}(\phi) = \phi$$
$$f^{-1}(Y) = X$$
$$f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$$
$$f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$$
$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$$

Directed graphs & equivalence relations

representation for relations over finite sets.

$R \subseteq X \times X$



directed graph.

(b, c)
(b, d)
(a, d)
(a, a)

relation.

$R \subseteq X \times X$ is an equivalence relation if it is

① reflexive: $\forall x,\ xRx$,

② symmetric: $\forall x,y \in X \quad xRy \Rightarrow yRx$

③ transitive: $\forall x,y,z \in X,\quad xRy$ and $yRz \Rightarrow xRz$

R_1 empty

a b

② T ③ T

① F

R_2

minimal equivalence

P is a partition of X iff P consists of non-empty subsets of X, s.t.

① $\bigcup_{p \in P} = X$

② $\forall\, p_1$ and p_2 in P, $p_1 \cap p_2 = \emptyset$

$\{x\}_R$

$X = \{ \cdots \mid \cdots \mid \cdots \mid \cdots \}$

Thm: Let R be a relation on X.

$X/R := \{ \{x\}_R \mid x \in X \}$: the set of equivalence classes w.r.t. R is a partition of X.

$\{x\}_R = \{y\}_R$ iff xRy.

proof: a) $\{x\}_R$ is non-empty because R is reflexive.

b) $\{x\}_R \cap \{y\}_R \neq \emptyset \Rightarrow xRy$.

and c) $xRy \Rightarrow \{x\}_R = \{y\}_R$.

For b): if $\{x\}_R \cap \{y\}_R \neq \emptyset$, then $z \in \{x\}_R$, $z \in \{y\}_R$

$\Rightarrow zRx$, zRy

$\xRightarrow{\text{transitivity of } R}$ xRy

For c) $\{x\}_R \subseteq \{y\}_R$

Take any $w \in \{x\}_R$, then wRx.

Because xRy, by tran of R, wRy.

Then $w \in \{y\}_R$

and $\{y\}_R \subseteq \{x\}_R$

Now we proved $\{x\}_R = \{y\}_R$ iff xRy.

Proposition: Let P be a partition of X.

We can define a relation R on X by

$$xRy \iff \exists p \in P, \text{ s.t. } x \in p \text{ and } y \in p.$$

R is an equivalence relation with $X/_R = P$.

Examples of equivalence relations.

$\mathbb{N}$

$a \equiv b \pmod{k}$ iff $a-b$ is divisible by k.

$k=3$

$$\mathbb{N}/_{(\text{mod } 3)} = \{ \underset{\{0,3,6\cdots\}}{\{0\}_{(\text{mod } 3)}}, \underset{\{1,4,7\}}{\{1\}_{(\text{mod } 3)}}, \underset{\{2,5,8\cdots\}}{\{2\}_{(\text{mod } 3)}} \}$$

Partial order on a set.

Def: A partial order is a set P on which there is a relation $\leq$ s.t.

① reflexive. $\forall p \in P$, $p \leq p$.

② transitive. $\forall p, q, r \in P$, $p \leq q$ & $q \leq r \Rightarrow p \leq r$.

③ antisymmetric. $\forall p, q \in P$ $p \leq q$ & $q \leq p \Rightarrow p = q$.

A total order is a partial order s.t. for any pair p, q, these elements are comparable,

i.e. $p \leq q$ or $q \leq p$.

Let P be the collection of subsets in S.

$(P, \subseteq)$ is a partial order.

Hasse diagram of $(P, \subseteq)$ where $P = 2^{\{a,b,c\}}$

Any partial order $(P, \leq)$ For $X \subseteq P$

(least upper bound): u s.t.
supremum
① it is an upper bound i.e. $\forall x \in X,\ x \leq u$
② it is least i.e. $(\forall x \in X,\ x \leq p) \Rightarrow u \leq p$

(greatest lower bound)
infimum
l s.t.
① $\forall x \in X,\ l \leq x$
② $(\forall x \in X,\ p \leq x) \Rightarrow p \leq l$

Both l.u.b and g.l.b are unique (if exists)

proof: u_1 and u_2 are both l.u.b. Then by ② $u_1 \leq u_2$ and $u_2 \leq u_1$.
By a.s. of partial order, $u_1 = u_2$.