Cardinality

Size of set

"wrong idea" Collection of set / R.

ARB : $\exists$ bij $f : A \to B$

A and B, if ARB are said to have the same size (cardinality).

Countable v.s. uncountable.

A set A is finite if $\exists$ bijection (bij) from the set $\{ m \in \mathbb{N} \mid m \le n \}$ to A, for some $n \in \mathbb{N}_0$

$\mathbb{N} = \{ 1, 2, \cdots \}$

A set A is infinite if it is not finite.

...... is countable iff it is finite or there is a bij $f : \mathbb{N} \to A$.

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{A}$ are

$\mathbb{R}$ is uncou

Lemma 1: Any subset of natural numbers is countable.

If $A \neq \phi$, then done.

Pf: Denote this subset by A.

Define a partial function $f : \mathbb{N} \to A$ by

① $f(1)$: least element in A.

② $f(n+1)$ if $f(n)$ is defined and let $f(n+1)$ to be least number i and is larger than f

If $f(n+1)$ is defined, then $f(n)$ is d

$D = \{ n \in \mathbb{N} \mid f(n) \text{ is defined} \}$.

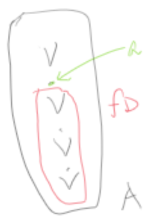
then D is either $\mathbb{N}$ or $\{ 1, 2, 3 \cdots n \}$.

Claim: $f: D \to A$ is bijective.

inj ↙ ↘ surj

① inj is easier. Strict order is preserved.

② surj: Assume for contradiction that $f: D \to A$ is not surjective.



Then $fD \subsetneq A$, or $A \setminus fD = \emptyset$

Take the least element $a \in A \setminus fD$.

① $a \le b$, $b \in fD$, because

$f(1) \in fD$

$f(1)$ is the smallest in f).

② We can find $a' < a$, s.t. a' is the largest element in fD, and $f(k) = a'$

Now by definition, $f(k+1) = a$.

Now a contradiction against $a \in A \setminus fD$.

$\Rightarrow$ f has to be surjective.

Corollary: A set B is countable iff there is a bij. $g: A \to B$ from some subset $A \subseteq \mathbb{N}$.

Pf: only if ($\Rightarrow$): by definition.

if ($\Leftarrow$): By lemma I:

initial segment of $\mathbb{N}$

$$\begin{tikzcd} D \arrow[r, "f \text{ (bij.)}"] \arrow[d, "g \circ f \text{ (bij.)}"'] & A \arrow[dl, "g \text{ (bij.)}"] \\ B & \end{tikzcd}$$

D is either $\{n \in \mathbb{N} \mid m < n\}$ or $\mathbb{N}$

$\exists$ bij $g \circ f: D \to B$. By def, B is countable.

Lemma 2:

A set B is countable iff there is an injection $f: B \to \mathbb{N}$.

proof: only if $\Rightarrow$

subset of $\mathbb{N}$

$$\begin{tikzcd} A \arrow[r, "g \text{ (bij.)}"] \arrow[d, "i"', "\text{inclusion (inj)}"] & B \arrow[l, bend left, "g^{-1} \text{ (bij.)}"] \arrow[dl, "i \circ g^{-1}"] \\ \mathbb{N} & \end{tikzcd}$$

$i \circ g^{-1}$ injective.

if $\Leftarrow$

$f: B \to fB$

$$\begin{tikzcd} B \arrow[r] \arrow[dr, "f"'] & fB \arrow[l, bend right, "f^{-1}: fB \to B \text{ (bij.)}"'] \arrow[d, "\text{inclusion}"] \\ & \mathbb{N} \end{tikzcd}$$

fB: subset of $\mathbb{N}$.

Any injection, when restricted to its image, is a bijection

Lemma 3: Set B is countable iff there is an injection

$f: B \to A$, A is countable.

Pf: only if $\Longrightarrow$ We can choose $A = \mathbb{N}$, by Lemma 2, we are done.

if $\Longleftarrow$

inj, countable

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[dr] & A \arrow[d, "g"] \\ & \mathbb{N} \end{tikzcd}$$

inj $\to$ Lemma 2

inj $\leftarrow$ $g \circ f$

$\exists$ inj $g \circ f : B \to \mathbb{N}$. Then by Lemma 2, B is countable.

proposition 1: $\mathbb{N} \times \mathbb{N}$ is countable.

Pf: Define. $\mathbb{N} \times \mathbb{N} \xrightarrow{f} \mathbb{N}$

$$(m, n) \xmapsto{f} 2^m \times 3^n.$$

f is injective.

By. Lemma 3, $\mathbb{N} \times \mathbb{N}$ is inj.

prop I': $\mathbb{Q}^+$ positive rational numbers is countable.

Pf: $\mathbb{Q}^+ \xrightarrow{\text{inj}} \mathbb{N} \times \mathbb{N}$ → countable by prop I

$$\frac{m}{n} \longrightarrow (m, n)$$

m, n. no common factor

By Lemma 3, $\mathbb{Q}^+$ is countable.

prop 2: A, B are countable, then $A \times B$ is countable.

pf:

$$A \xrightarrow{f_A} \mathbb{N}$$

$$B \xrightarrow{f_B} \mathbb{N}$$

both inj

$$A \times B \xrightarrow{f_A \times f_B} \mathbb{N} \times \mathbb{N}$$

inj

$$a, b \longmapsto (f_A(a), f_B(b))$$

prop 3: $A_1 \cup A_2 \cup \cdots \cup A_k$

and $A = \bigcup_{n \in \mathbb{N}} A_n := \{ x \mid \exists n \in \mathbb{N},\ x \in A_n \}$

are countable if all A_n's are countable.

prop 4: $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

$$\mathbb{Z} = \mathbb{Z}^{-} \cup \{0\} \cup \mathbb{Z}^{+}$$

$\mathbb{Z}^{-} \to \mathbb{N}$, $\mathbb{Z}^{+}$: $\mathbb{N}$

$$\mathbb{Q} = \underbrace{\mathbb{Q}^{-}}_{\text{countable}} \cup \{0\} \cup \underbrace{\mathbb{Q}^{+}}_{\text{countable}}$$

Pf: $\forall n \in \mathbb{N}$, $\exists f_n : A_n \to \mathbb{N}$ inj.

Define. $A \longrightarrow \mathbb{N} \times \mathbb{N}$ injective

index

$x \longmapsto (n_x, f_{n_x}(x))$

$x \in A$, let n_x be the least number s.t. $x \in A_{n_x}$.

[Cantor 1874] $\mathbb{R}$ is uncountable.

Pf: Assume $\mathbb{R}$ is countable.

Then $(0,1] = \{ r \in \mathbb{R} \mid 0 < r \leq 1 \}$ is also countable. $\boxed{\text{Lemma 3}}$

$(0,1]$ is an infinite set.

There is a bij $\boxed{f}$ from $\mathbb{N}$ to $(0,1]$

$$f: \mathbb{N} \to (0,1]$$

$$f(n) = 0.d_1^n d_2^n d_3^n \cdots\cdots$$

Key: Define a number in $(0,1]$

$$R = 0.r_1 r_2 r_3 \cdots\cdots$$

s.t. $r_i = \begin{cases} 1 & \text{if } d_i^i \neq 1 \\ 2 & \text{if } d_i^i = 1 \end{cases}$

$R \in (0,1]$

Now a contradiction: f is bij.

$$\exists k \in \mathbb{N}, \ f(k) = R$$

$$f(k) = 0.\, d_1^k\, d_2^k \cdots \cdots \boxed{d_k^k} \cdots \cdots \cdots$$

$$\| \qquad\qquad \updownarrow$$

$$R = 0.\, r_1\, r_2 \cdots \cdots r_k \cdots \cdots$$

if $r_k = d_k^k = 1$, by def. of R, $r_k = 2 \neq 1$

if $r_k = d_k^k \neq 1$ by - — - - $r_k = 1$