Real number axioms

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continuous line segment. has more points than discrete sequence

$|[a,b]| > |\{r_i\}_{i=0}^{\infty}|$ (prove this)

$[a_0, b_0]$

Argument: Assume they are 1-1 corresponding to each other.

step 1: find the 1st pair of numbers in $\{r_i\}_{i=0}^{\infty}$ s.t. the pair is in $[a,b]$.

step 2: name the image of pair in step 1 $[a_1, b_1]$

$[a_1, b_1] \subseteq [a,b]$.

Repeat step 1: find a pair in $\{r_i\}_{i=0}^{\infty}$ that's in $[a_1, b_1]$

⋮

step n: $[a_{n-1}, b_{n-1}]$ repeat,

One get $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$

⋮

$\bigcap_{k=1}^{\infty} [a_k, b_k] \neq \emptyset \Rightarrow$ Cantor Thm

Is $\mathbb{R}$ or $[a,b]$ complete

"∃ hole?"

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7. completeness axioms

① (least upper bound property)

Every non-empty subset of $\mathbb{R}$, if it has an upper bound, then it has a unique least upper bound.

______, ______ lower.

greatest lower bound.

$\sup\left(S=\{x\in\mathbb{Q} \mid x^2<2\}\right)=\sqrt{2}.$

② Dedekind's cut. Idea: complete $\mathbb{Q}$

We can cut $\mathbb{R}$ at a point x.

① $x\in\mathbb{Q}$.

② $x\notin\mathbb{Q}$, then it can be named using a sequence of rational numbers.

E.g. $\{x \mid x^2>2 \text{ and } x>0\}$ $\{x \mid x^2<2 \text{ or } x$

this cut defines irrational number $\sqrt{2}$.

③ $\mathbb{R}$ is Cauchy complete.

Def: (Cauchy sequence) $\{a_i\}_{i=1}^{\infty}$

$<0 \}$

$\forall \varepsilon > 0,$

$\exists N \in \mathbb{N}$, s.t. $\forall m, n > N$

Cauchy sequence in $\mathbb{Q}$ may

Take all limit of Cauchy sequence

name them a number,

completion

$\mathbb{R}$ is Cauchy complete.

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④ Nested interval Theorem

$I_n = [a_n, b_n]$ s.t.

$I_1 \supset I_2 \supset I_3 \supset \cdots$ I

$N \quad |a_m - a_n| < \varepsilon$

not coverge in $\mathbb{Q}$.

e in $\mathbb{Q}$,

call the

$\mathbb{R}$.

$I_1 \supset I_2 \supset I_3 \supset \dots I$

Moreover, $(b_n - a_n) \to 0$

$\bigcap_{n=1}^{\infty} I_n$. contains exa

$\sqrt{2} = 1.4142358\dots$

$[1, 2] \supset [1.4, 1.5]$

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5 monotone convergence T

$\mid \; \mid \; \}^{\infty}$ mon

$n \supset \cdots$

as $n \to \infty$.

ctly one point.

$\supset [1.41, 1.42] \supset \cdots$

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hm.

$\{ b_i \}_{i=1}^{\infty}$ non-d

6 Bolzano Weierstrass

Every bounded seq

7 Intermediate value t

1 $\Rightarrow$ 4 $\Rightarrow$

ecreasing, bounded sequence,
it has a finite number as its

uence has a converging subsequ

heorem.

6

limit.

enl