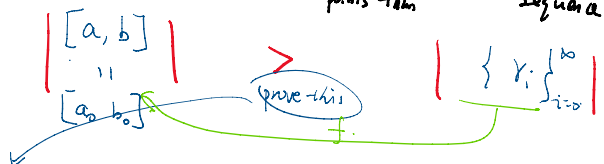


Continuous line segment. has more points than discrete sequence



Argument: Assume they are 1-1 corresponding to each other.

step 1: find the 1st pair of numbers in  $\{r_i\}_{i=0}^n$  s.t. the pair is in  $[a, b]$ .

step 2: name the image of pair in step 1  $[a_1, b_1] \subseteq [a, b]$

Repeat step 1: find a pair in  $\{r_i\}_{i=0}^\infty$  that's in  $[a_1, b_1]$

step n:  $[a_{n-1}, b_{n-1}]$  repeat.

one set  $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$

$$\bigcap_{k=1}^{\infty} [a_k, b_k] \neq \emptyset \Rightarrow \text{Cantor Thm}$$

Is  $\mathbb{R}$  or  $[a, b]$  complete

↓  
"hole?"

## 7. completeness axioms

① (least upper bound property)

Every non-empty subset of  $\mathbb{R}$ , if it has an upper bound, then it has a unique least upper bound.



lower.





\_\_\_\_\_ ,  
\_\_\_\_\_ → greatest lower bound.

$$\sup \left( S = \{ x \in \mathbb{Q} \mid x^2 < 2 \} \right) = \sqrt{2}$$

② Dedekind's cut. Idea: complete  $\mathbb{Q}$

We can cut  $\mathbb{R}$  at a point  $x$ .

①  $x \in \mathbb{Q}$ .

②  $x \notin \mathbb{Q}$ , then it can be named using a sequence of rational numbers.

E.g.  $\{ x \mid x^2 > 2 \text{ and } x > 0 \}$  —  $\{ x \mid x^2 < 2 \text{ or } x < 0 \}$   
this cut defines irrational number  $\sqrt{2}$ .

③  $\mathbb{R}$  is Cauchy complete.

Def: (Cauchy sequence)  $\{ a_i \}_{i=1}^{\infty}$

$\int_0^1 x^2 dx$

$\int_0^1 x^2 dx$



$\exists N \in \mathbb{N}$ , st.  $\forall m, n > N$

Cauchy sequence in  $\mathbb{Q}$  may

Take all limit of Cauchy sequences

name them a number,  
completion

$\mathbb{R}$  is Cauchy complete.

---

④

Nested interval Theorem

$I_n = [a_n, b_n]$  st.

$I_1 \supset I_2 \supset I_3 \supset \dots$

$$\forall |a_m - a_n| < \epsilon$$

not converge in  $\mathbb{Q}$ .

is  $\mathbb{Q}$ ,

call the

$\mathbb{R}$ .

---

$\dots$





$$I_1 \supset I_2 \supset I_3 \supset \dots I_n$$

Moreover,  $(b_n - a_n) \rightarrow 0$

$\bigcap_{n=1}^{\infty} I_n$  contains exactly

$$\sqrt{2} = 1.41421356\dots$$

$$[1, 2] \supset [1.4, 1.5]$$

(5)

monotone convergence T

1 1  $\rightarrow \infty$

$n \rightarrow \infty$

$n > \dots$

as  $n \rightarrow \infty$ .

exactly one point.

$> [1.41, 1.42] > \dots$

---

them.

Increasing  $1, 1, 1, \dots$



$$\{ b_i \}_{i=1}^{\infty}$$

non-d

6

Bolzano Weierstrass

Every bounded seq

7

Intermediate value t

1

$\Rightarrow$

4

$\Rightarrow$

(

decreasing, bounded sequence,

it has a finite number as its

limit has a converging subsequence

theorem.

6

limit.

ence.