Set notation vs lambda notation

Thursday, February 23, 2023 11:10 AM

Exercise 4.2 Let X and Y be sets. Define the *projections*

$$\pi_1 : X \times Y \to X \text{ and } \pi_2 : X \times Y \to Y$$

by taking $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$ for $(a,b) \in X \times Y$.

Let Z be a set and $f : Z \to X$ and $g : Z \to Y$. Show that there is a unique function $h : Z \to X \times Y$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$.

$\exists!$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \\ & Z \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, dashed, "h"] & \end{tikzcd}$$

Existence:

we construct h as follows

$$z \in Z \xmapsto{h} (f(z),\ g(z))$$

check: $\pi_1 \circ h(z) = f(z)$ done.

$\pi_2 \circ h(z) = g(z)$.

Uniqueness: Assume $\exists\ k : Z \to X \times Y$ st $\pi_1 k = f$ and $\pi_2 k = g$

$$k(z) = (x_k,\ y_k)$$

$$\pi_1 . k(z) = \pi_1(x_k, y_k) = x_k = f(z)$$

$$\pi_2\, k(z) = \pi_2(x_k, y_k) = y_k = g(z)$$

Two conditions $\Rightarrow$ $k(z) = (f(z), g(z))$.

Disjoint union

Frequently we want to join sets together but, in a way which, unlike union, does not identify the same element when it comes from different sets. We do this by making copies of the elements so that when they are copies from different sets they are forced to be distinct:

$$X_1 \uplus X_2 \uplus \cdots \uplus X_n = (\{1\} \times X_1) \cup (\{2\} \times X_2) \cup \cdots \cup (\{n\} \times X_n).$$

In particular, for $X \uplus Y$ the copies $(\{1\} \times X)$ and $(\{2\} \times Y)$ have to be disjoint, in the sense that

$$(\{1\} \times X) \cap (\{2\} \times Y) = \emptyset,$$

because any common element would be a pair with first element both equal to 1 and 2, clearly impossible.

Exercise 4.3 Let X and Y be sets. Define the *injections*

$$inj_1 : X \to X \uplus Y \text{ and } inj_2 : Y \to X \uplus Y$$

by taking $inj_1(a) = (1, a)$ for $a \in X$, and $inj_2(b) = (2, b)$ for $b \in Y$.

Let Z be a set and $f : X \to Z$ and $g : Y \to Z$. Show that there is a unique function $h : X \uplus Y \to Z$ such that $h \circ inj_1 = f$ and $h \circ inj_2 = g$.

$$\begin{tikzcd} & X \uplus Y \arrow[dd, dashed, "h"] & \\ X \arrow[ur, "inj_1"] \arrow[dr] & & Y \arrow[ul, "inj_2"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

$X \uplus Y =$

existence $(\{1\} \times X) \cup (\{2\} \times Y)$

$$h(\overset{X \uplus Y}{\overset{\cup}{(n, e)}}) = \begin{cases} f(e) & \text{if } (n,e) \in \{1\} \times X \\ g(e) & \text{if } (n,e) \in \{2\} \times Y \end{cases}$$

uniqueness :

If $\exists\, k$ s.t. $k \circ inj_1 = f$, $k \circ inj_2 = g$.

then $k = h$

$k|_{\{1\} \times X} = h$ because $k \circ inj_1 = f = h \circ inj_1$

$\Leftarrow$ $\boxed{u \circ inj = v \circ inj \Rightarrow u = v}$

$k|_{\{2\} \times Y} = h$ because $k \circ inj_2 = g = h \circ inj_2$

$\lambda\, a \in A \,.\, e$ what it does

Exercise 4.8 Use lambda notation to describe bijections.

$$[(A \times B) \to C] \cong [A \to (B \to C)]\ ,$$
$$[A \to (B \to C)] \cong [B \to (A \to C)]\ .$$

$g(a, b) \to C$

□

$[(A \times B) \xrightarrow{g} C] \mapsto [A \to (B \to C)]$

$\lambda g . (\lambda a . (\lambda b . g(a,b)))$

$[(A \times B) \to C] \longleftarrow [A \xrightarrow{f} (B \to C)]$

$\boxed{f(a)} \to (B \to C)$

$\lambda f . (\lambda (a,b) . (f(a)(b)))$

$f(a)(b) \in C$

□

Exercise 4.9 Describe explicit bijections

$$[(A \uplus B) \to C] \cong (A \to C) \times (B \to C),$$
$$[A \to (B \times C)] \cong (A \to B) \times (A \to C)\ .$$