Inductive definitions

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set defined by rules

premises $\leftarrow$ $\dfrac{x_1, x_2 \ \ldots}{y}$

conclusion $\leftarrow$

A rule is called finitary if it has finite premises.

$\dfrac{\quad}{a}$ $\rightarrow$ this kind of rule is call an axiom

E.g.1.

Define natural numbers

1st way: $\dfrac{\quad}{0}$ $\qquad \dfrac{n}{n+1}$

2nd way: $\dfrac{\quad}{0}$, $\qquad \dfrac{0, 1 \ \ldots \ (n-1)}{n}$

Eg. 2. set of strings Σ^* over alphabet Σ.

finite list of symbols

collection of symbols

$$\Sigma^* = \quad \frac{}{\varepsilon} \qquad \frac{x}{ax} \quad a \in \Sigma.$$

empty string.

concatenation.

Inductively-defined sets

$\frac{1}{2} \to 1/2$.

$(X/y) \to$ rule

$X = \{x_1 \cdots x_k \cdots\}$ premises

y conclusion

$(\emptyset/y) \to$ axiom

$$\mathbb{N}_0 = \{ (\emptyset/0) \cup (\{n\}/n+1) \}$$

$$\Sigma^* = \{ (\emptyset/\varepsilon) \cup (\{x\}/ax) \ \forall a \in \Sigma \}$$

A set of rule specifies a way to define a set.

Let R be a set of rules, a set Q is closed under R if for all rules (X/y)

$R = \{ (\cancel{X}/y) \cdots \}$

$$X \subseteq Q \Rightarrow y \in Q.$$

Rmk: A R-closed set Q must contain all the axioms.

All R-closed sets:

$$\boxed{\{ Q \mid Q \text{ is R-closed} \}}$$

is non-empty.

because $\{ y \mid \exists X.\ (X/y) \in R \}$ large

A smaller R-closed set:

$$I_R = \bigcap \{ Q \mid Q \text{ is R-closed} \}$$

proposition: ① I_R is R-closed

② if Q is R-closed, then $\boxed{I_R \subseteq Q}$.

proof: ① let $(X/y) \in R$, suppose $X \subseteq I_R$.

Pick ~~a~~ arbitrary R-closed set Q, then $X \subseteq Q$ by def.

Because Q is R-closed, $y \in Q$. Hence I_R is R-closed.

② ✓ by definition.

Principle of rule induction:

$P(x)$ property or proposition.

We wish to show $P(x)$ is T for all $x \in I_R \iff Q = I_R$

$$Q = \{ x \in I_R \mid P(x) \} \subseteq I_R.$$

we only need $I_R \subseteq Q$

By prop ②, we only need Q is R-closed.

for all $(X/y) \in R$

$$(\forall x \in X.\ x \in I_R \ \&\ P(x)) \Rightarrow (y \in I_R \ \&\ P(y))$$

Because I_R is R-closed,

$\forall x \in X.\ x \in I_R \Rightarrow y \in I_R$ follows.

then $\boxed{(\forall x \in X.\ x \in I_R \ \&\ P(x)) \Rightarrow P(y)}$

principle of rule induction.

Exercise 5.8 The set S is defined to be the least subset of natural numbers $\mathbb{N}$ such that:

$1 \in S$;

if $n \in S$, then $3n \in S$;

if $n \in S$ and $n > 2$, then $(n-2) \in S$.

Show that $S = \{m \in \mathbb{N} \mid \exists r, s \in \mathbb{N} \cup \{0\}. \quad m = 3^r - 2s\}$. Deduce that S is the set of odd numbers. $\square$