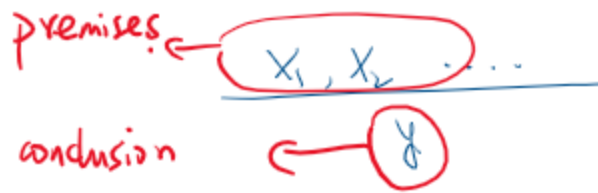
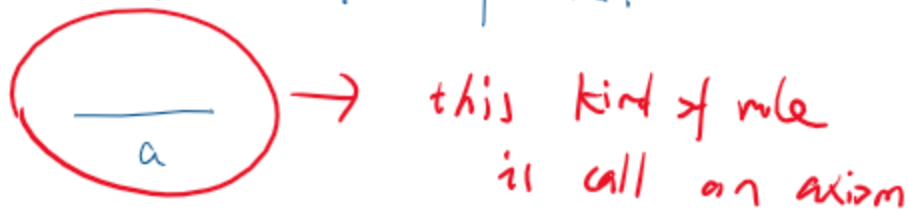


set defined by rules



A rule is called finitary if it has finite premises.



E.g. 1.

Define natural numbers

1st way: $\frac{}{0}$ $\frac{n}{n+1}$

2nd way: $\frac{}{0}$, $\frac{0, 1, \dots, (n-1)}{n}$

eg. 2. set of strings Σ^* over alphabet Σ
 (finite list of symbols)
 collection of symbols

$\Sigma^* = \{ \epsilon \} \cup \{ ax \mid a \in \Sigma, x \in \Sigma^* \}$
 empty string ϵ
 concatenation

Inductively-defined sets

$\frac{1}{2} \rightarrow \frac{1}{2}$

$(X/y) \rightarrow$ rule

$X = \{ x_1, \dots, x_n, \dots \}$ premises

y conclusion

$(\phi/y) \rightarrow$ axiom

$\mathbb{N}_0 = \{ (\phi/0) \cup \{n\}/n+1 \}$

$\Sigma^* = \{ (\phi/\epsilon) \cup \{x\}/ax \mid a \in \Sigma \}$

A set of rules specifies a way to define a set.

Let R be a set of rules, a set Q is closed

$\{ (X/y) \dots \}$

under R if, for all

rules (X/y)

$X \subseteq Q \Rightarrow y \in Q$.

Rule: A R -closed set Q must contain all the axioms.

All R -closed sets:

$$\{ Q \mid Q \text{ is } R\text{-closed} \}$$

is non-empty.

because $\{ y \mid \exists x. (x/y) \in R \}$ large

A smaller R -closed set:

$$I_R = \bigcap \{ Q \mid Q \text{ is } R\text{-closed} \}$$

Proposition: ① I_R is R -closed

② if Q is R -closed, then $I_R \subseteq Q$.

proof: ① let $(x/y) \in R$, suppose $x \in I_R$.
Pick an R -closed set Q , then $x \in Q$ by def.
Because Q is R -closed, $y \in Q$. Hence I_R is R -closed.
② \forall by definition.

Principle of rule induction:

$P(x)$ property or proposition.

We wish to show $P(x)$ is \top for all $x \in I_R$ \Leftrightarrow $Q = I_R$

$$Q = \{ x \in I_R \mid P(x) \} \subseteq I_R.$$

we only need $I_R \in Q$

By prop (2), we only need Q is R -closed.
for all $(X/y) \in R$.

$$(\forall x \in X. x \in I_R \ \& \ P(x)) \Rightarrow (\underline{y \in I_R} \ \& \ P(y))$$

Because I_R is R -closed,

$\forall x \in X. x \in I_R \Rightarrow y \in I_R$ follows.

then \Downarrow

$$(\forall x \in X. x \in I_R \ \& \ P(x)) \Rightarrow P(y)$$

principle of rule induction.

Exercise 5.8 The set S is defined to be the least subset of natural numbers \mathbb{N} such that:

$$1 \in S;$$

$$\text{if } n \in S, \text{ then } 3n \in S;$$

$$\text{if } n \in S \text{ and } n > 2, \text{ then } (n - 2) \in S.$$

Show that $S = \{m \in \mathbb{N} \mid \exists r, s \in \mathbb{N} \cup \{0\}. \quad m = 3^r - 2s\}$. Deduce that S is the set of odd numbers. \square