Math 433 W 2019 - Exam 2- Apr 3, 11:00(sharp)-11:50 -
Name:

There are 4 pages and 3 questions in Exam 2. You are allowed to use at most one 3-inch by 5-inch two-sided note card.
Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics. Partial credits will be given for partially correct arguments.

1.[50 pts] We know that in x-y-z space, the graph of a differentiable function $z = f(x, y)$ with domain $\mathbb{R}^2$ is a regular surface, whose parametrization is $\sigma(x, y) = (x, y, f(x, y))$. Now we consider the elliptic paraboloid E given by the graph of

$$z = \frac{1}{2}(x^2 + y^2), \quad \forall x, y \in \mathbb{R}.$$

a) [10 points] Verify that $I_E = (1+x^2)dx^2 + 2xydxdy + (1+y^2)dy^2$. (You can start with either the definition or any formula we covered, please write down key steps.)

Solution Parametrization: $\sigma(x, y) = (x, y, \frac{1}{2}(x^2 + y^2))$. Hence

$$\sigma_x = (1, 0, x), \quad \sigma_y = (0, 1, y).$$

Then

$$E = \sigma_x \cdot \sigma_x = 1 + x^2 \quad F = \sigma_y \cdot \sigma_x = xy \quad G = \sigma_y \cdot \sigma_y = 1 + y^2.$$

b) [10 points] Verify that $II_E = \frac{1}{\sqrt{1+x^2+y^2}}dx^2 + \frac{1}{\sqrt{1+x^2+y^2}}dy^2$. (Please write down key steps.)

Solution

$$\sigma_{xx} = (0, 0, 1), \quad \sigma_{yy} = (0, 0, 1) \quad \sigma_{xy} = (0, 0, 0).$$

And we also have

$$\sigma_x \wedge \sigma_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & x \\ 0 & 1 & y \end{vmatrix}$$

$$\vec{n} = \frac{\sigma_x \wedge \sigma_y}{||\sigma_x \wedge \sigma_y||} = \frac{(-x, -y, 1)}{\sqrt{1 + x^2 + y^2}}.$$

Then

$$L = \sigma_{xx} \cdot \vec{n} = \frac{1}{\sqrt{1 + x^2 + y^2}} \quad L = \sigma_{xy} \cdot \vec{n} = 0 \quad N = \sigma_{yy} \cdot \vec{n} = \frac{1}{\sqrt{1 + x^2 + y^2}}.$$

c) [20 points] At each point $(x, y, z) \in E$, the Gaussian curvature K is a function $K(x, y)$, only depends on x, y. Work out the formula for $K(x, y)$. And use it to compute the Gaussian curvature at point $(0, 0, 0)$ and $(1, 1, 1)$.

Solution

$$K = \frac{LN - M^2}{EG - F^2} = \frac{1}{(1 + x^2 + y^2)^2}.$$

$$K(0, 0, 0) = 1; K(1, 1, 1) = \frac{1}{9}.$$

d)[10 points] Based on part c), is there an isometry $f : R \to S$, from the open region

$$R = \{(x, y, z) \in \mathbb{R}^3 | z = \frac{1}{2}(x^2 + y^2), \quad -2 < x, y < 2\}$$

in E to some open subset S in the sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 9\}?$$

(You can start with any fact we know about the sphere, make sure to give a brief explanation of your answer.)

Solution No. Because the sphere has constant Gaussian curvature. In this case, $K_{S^2} \equiv \frac{1}{9}$. Since K is invariant under isometry by Gauss's Theorema Egregium, there's no isometry from the region R with non-constant K to any open subset $S \subset S^2$ with constant Gaussian curvature.

2. [25 pts] a)[10 points] We know the Euclidean (u, v)-plane $\mathbb{R}^2$ has $I = du^2 + dv^2$ and $II = 0$. Answer the following questions with brief explanations (either geometric or computational).

- What's the Gaussian curvature at any point $p \in \mathbb{R}^2$?
- What's the normal curvature at any point $p \in \mathbb{R}^2$ along the direction $\vec{v} = (a, b)$?

Solution At any point,

$$K = \frac{LN - M^2}{EG - F^2} = 0.$$

For the normal curvature at any, one can choose the straight line along direction $\vec{v} = (a, b)$. Since any straight line has curvature 0, by $k^2 = k_n^2 + k_g^2$, any normal curvature is 0.

Or we can consider 2×2 matrix $A_1 = \boldsymbol{Id}, A_2 = \mathbf{0}$, we compute principle curvature by solving $det(\mathbf{0} - k\boldsymbol{Id}) = 0$, which means $k = 0$. Then every point is umbilical and principle curvature is 0. By Euler's Theorem, $k_n \equiv 0$.

b) [15 points] We know that local isometry preserves the first fundamental form. But it might change the second fundamental form, for example, the cylinder parameterized by $\alpha(u, v) = (\cos u, \sin u, v)$ has $I = du^2 + dv^2, II = (-1)du^2$.

Now question is can we find a surface $\sigma(u, v)$, with $I = du^2 + dv^2$ and $II = (-1)du^2 + dv^2$? **Hint:** Gaussian curvature and Gauss Theorema Egregium should be enough. Of course, Gauss's equation overkills the question, but that's a bit too complicated.

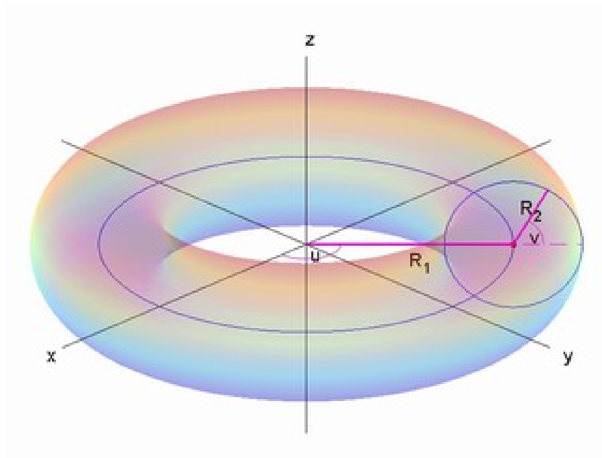
Solution If such surface exist, on the one hand,

$$K = \frac{LN - M^2}{EG - F^2} = -1;$$

on the other hand, the surface is locally isometric to the plane since they have the same I, where follows from Gauss Theorema Egregium $K = 0$. Contradiction. Hence such surface does not exist.

3.[25 pts]As in the following picture, a torus is parameterized by $\Sigma(u,v) = ((R_1+R_2\cos v)\cos u, (R_1+R_2\cos v)\sin u, R_2\sin v)$. In general, we have

$$I = (R_1 + R_2\cos v)^2 du^2 + R_2^2 dv^2, \quad II = \cos v(R_1 + R_2\cos v)du^2 + R_2 dv^2.$$



a) [10 points] Let $R_1 = 100, R_2 = 1$, i.e. $\sigma(u,v) = ((100+\cos v)\cos u, (100+\cos v)\sin u, \sin v), u \in [0, 2\pi], v \in [0, 2\pi]$. What's the Gaussian curvature $K(u,v)$ at the point $\sigma(u,v)$? **Hint:** This is our familiar case, torus as a surface of revolution, obtained by rotating the arc-length parameterized curve $C = (100 + \cos v, \sin v)$ along the z-axis.

Solution Since we have a surface of revolution with arc-length parameterized curve $C = (f(v), g(v)) = (100 + \cos v, \sin v)$,

$$K = -\frac{f_{vv}}{f} = \frac{\cos v}{100 + \cos v}.$$

Or you can do direction computation using the definition.

b)[10 pts] Fill in blanks with A, B, C, D: $A.(99, 0, 0)$ $B.(100, 0, 1)$ $C.(101, 0, 0)$ $D.$None of A, B, C.

The torus $\sigma(u,v) = ((100+\cos v)\cos u, (100+\cos v)\sin u, \sin v)$ has an elliptic point(where $K > 0$) at C, a hyperbolic point(where $K < 0$) at A, a parabolic point (exactly one principal curvature=0) at B, a planar point (both principal curvatures=0) at D, an umbilical point (two principal curvatures are equal) at D.

c) [5 points] What is the expression A in the area formula $\iint_{u,v\in[0,2\pi]} A du dv$ of $\sigma(u,v)$? Compute the total integral of the Gaussian curvature on the torus $\sigma(u,v)$, i.e. evaluate the definite integral $\iint_{u,v\in[0,2\pi]} K(u,v) A du dv$. **Hint:** there's some symmetry of the Gaussian curvature $K(u,v)$ on the torus that can completely reduce the calculation.

Solution The total integral of the Gaussian curvature means $\int_{\mathbb{T}^2} K \cdot dA$, where dA is the area element on the torus. And in local (u,v) coordinates

$$\iint_{u,v\in[0,2\pi]} K(u,v) \cdot A du dv,$$

where A is the area element in (u, v) coordinates, or you can think as a Jacobian, which equals $\sqrt{EG - F^2}$. Then one can do the following computation:

$$\iint_{u,v\in[0,2\pi]} K(u,v)\cdot A\,dudv = \iint_{u,v\in[0,2\pi]} \frac{\cos v}{100+\cos v}\cdot(100+\cos v)dudv$$
$$= 2\pi \int_{v\in[0,2\pi]} \cos v dv = 0.$$

The purpose of this question was trying to say that on the side($\frac{\pi}{2} < v < \frac{3\pi}{2}$) of the torus we have $K < 0$ and on the other side $K > 0$. The point on the $K < 0$ side has gaussian curvature whose absolute value is larger than the $K > 0$ side, meanwhile, the $K < 0$ side has smaller area than the $K > 0$ side. The two effects cancel each other, by the above computation, and hence the total Gaussian curvature is 0.

If you already know Gauss-Bonnet Theorem(which we are going to cover next weeks), then you know the answer is 0.