Math 433 W 2019 - Final Apr 29, 4:00(sharp)-6:00 pm. EH4088.

Name:

There are 6 questions in the final, totally counting 150 points. You are allowed to use at most one 3-inch by 5-inch two-sided note card. You can directly use any proved results from lectures and HWs. Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics. Partial credits will be given for partially correct arguments.

1.[30 pts] Consider the Helix γ on a cylinder S

$$\gamma(t) = (\sin t; \cos t; t) \subset S, \quad t \in \mathbb{R},$$

where S is given by equation $\{x^2 + y^2 = 1\} \subset \mathbb{R}^3$.

a.[10 pts] Find the arc-length parametrization of γ.

b.[15 pts] Calculate the **unit normal vector** $\vec{n}_\gamma$ and the **curvature** k at any point of γ.

c.[5 pts] What is the geodesic curvature of γ on the cylinder S at a given point $p \in \gamma$?

Solution: a.Parametrization by arc length: $\alpha(s) = (\sin(s/\sqrt{2}); \cos(s/\sqrt{2}); (s/\sqrt{2}))$. b.

$$\vec{t} = \alpha'(s) = \frac{1}{\sqrt{2}}(\cos(s/\sqrt{2}); -\sin(s/\sqrt{2}); 1).$$

$$\alpha''(s) = \frac{1}{2}(-\sin(s/\sqrt{2}); -\cos(s/\sqrt{2}); 0).$$

Then $k = \frac{1}{2}$ and $\vec{n}_\gamma = (-\sin(s/\sqrt{2}); -cos(s/\sqrt{2}); 0)$.

c.

$$\vec{n}_\gamma = \pm \vec{n}_S$$

or Helix is a geodesic on the cylinder, $k_g = 0$.

Grading Note: Correct approach with numerical error will receive at least 80% of total grades. But incorrect arc-length parametrization will have at least 4 points off. Negative curvature will have 2 off since standard curvature k is always non-negative. For the k_g, any correct definition or formula of k_g will receive 3 points. Saying geodesic has $k_g = 0$ will receive full credits.

2.True or False with brief explanation[5pts$\times$ 3]

Grading note: Incorrect answer will receive at most 2 pts, correct answer at least has 3 pts, and correct answer with complete explanation receive 5 pts

a. For any given smooth functions $k(s) > 0$ and $\tau(s)$, there exists an arc-length parameterized regular curve $\alpha(s) \subset \mathbb{R}^3$ with curvature $k(s)$ and torsion $\tau(s)$.

Solution: True. This statement is part of the Fundamental theorem of curves.

b. Suppose on a surface S in $\mathbb{R}^3$, any point $p \in S$ has Gaussian curvature $K(p) \neq 0$. Then, for a given point $q \in S$, any curve passing through this point has normal curvature $k_n \neq 0$.

Solution: False. Consider the hyperboloid H with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Geometrically one can tell everywhere the surface has $K < 0$. This means there is one positive principal curvature and one negative curvature. However, we can find lines on H by letting $x = a$. Those lines has normal curvature 0.

Another approach is to consider Euler's theorem $k_n = k_1 \cos\theta + k_2 \sin\theta$. Hence at any point with negative Gaussian curvature, we can find direction with $k_n = 0$.

c. Let S be a compact connected orientable surface that has everywhere positive Gaussian curvature. Suppose there are two simple closed geodesics $\gamma_1, \gamma_2 \subset S$, then they must intersect.

Solution: True. A complete explanation will contain the following key points:

1. By Gauss-Bonnet and classification theorem, a compact connected orientable surface that has everywhere positive Gaussian curvature is homoemorphic to the sphere.

2. If assume on a sphere two simple closed geodesics $\gamma_1 \cap \gamma_2 = \emptyset$, then they bound a cylinder S.

3. Hence a contraction $0 < \int_S K dA = \chi(S) = 0$.

3.[20 pts] Let $\sigma(u,v)$ be a surface parametrization where $u \geq 0, v \geq 0$, and suppose that the first fundamental form is

$$I_\sigma = \begin{pmatrix} u+v & \sqrt{u} \\ \sqrt{u} & 1 \end{pmatrix}.$$

a.[10 pts] Let $\gamma(t) = (t,1)$ be a regular curve on the (u,v)-plane, and let $\alpha(t) = \sigma \circ \gamma(t)$ be the curve on the surface $\sigma(u,v)$. Compute the arc-length of $\alpha(t) = \sigma \circ \gamma(t)$ for $t \in [0,3]$.

Solution: The arc-length element is $\sqrt{\vec{t}^T \cdot I_\sigma \cdot \vec{t}}$, where $\vec{t} = (1,0)$ is the tangent vector on the u,v-plane. And we have

$$\vec{t}^T \cdot I_\sigma \cdot \vec{t} = (1,0) \cdot \begin{pmatrix} t+1 & \sqrt{t} \\ \sqrt{t} & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then the arc-length of $\alpha(t) = \sigma \circ \gamma(t)$ is

$$\int_{[0,3]} \sqrt{t+1}dt = \frac{14}{3}.$$

b.[10 pts] Let $D = \{1 < u < 4; 1 < v < 4\}$ be the domain on (u,v)-plane, and let $\sigma(D)$ be the corresponding region on the surface. Calculate the area of $\sigma(D)$.

Solution: The arc-length element is $\sqrt{EG - F^2} = \sqrt{u+v-v} = \sqrt{v}$. Hence the area is

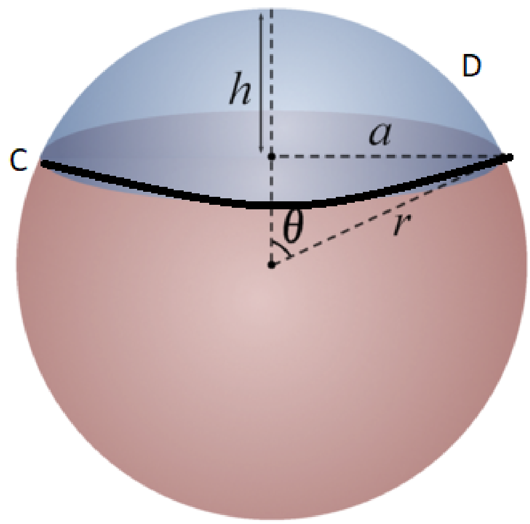
$$\iint_{[0,4]\times[0,4]} \sqrt{v}dudv = 3\int_{[0,4]} \sqrt{v}dv = 14.$$

Grading Note: The point of this question is the arc-length and area element. Any correct formula of the elements receive 60% more credits. Correct integration formula will have 80% or more. But integration with incorrect arc-length or area element will at most receive 7 points. Not taking square root of $\vec{t}^T \cdot I_\sigma \cdot \vec{t}$ or $EG - F^2$ will be 2 points off each.

4.[25 pts] Suppose the earth is a round sphere with radius one (we set the unit to be "one earth radius"). Now we consider the circle C which is the circle of Latitude $N30°$, and the spherical dome D which is the domain bounded by circle C that contains the north-pole. In the picture, D is the upper cap, and $r = 1, \theta = \frac{\pi}{3}$.

a.[10 pts] What is the geodesic curvature at a given point on the circle C?

b.[15 pts] Recall the fact that the Euler number of a homeomorphic disk(with boundary) is 1. Applying Gauss-Bonnet, what is the area of the spherical dome D? [**Hint:** You may use spherical coordinate to verify your answer.]



Solution: Through out this question we use the arc-length parametrization of C going counter-clockwise when looking down from the north; and choose the inward normal vector(which makes everything positive).

a. The circle C has radius $a = \sin\theta = \sqrt{3}/2$. This means the curvature $k = \frac{2}{\sqrt{3}}$. And the angle between the curve normal and the surface normal is $\frac{\pi}{2} - \theta$. Then $k_g = k\sin(\frac{\pi}{2} - \theta) = \frac{1}{\sqrt{3}}$.

b. Apply Gauss-Bonnet to the dome D with boundary C. We have

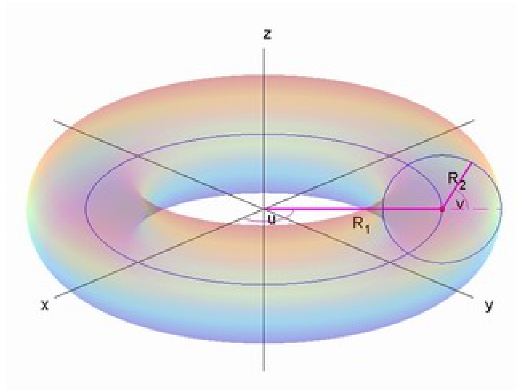
$$\iint_D K dA + \int_C k_g ds = 2\pi\chi(D) = 2\pi.$$

Since this is the unit sphere we have $K \equiv 1$, and hence $\iint_D K dA = area(D)$. From part a, we have $\int_C k_g ds = 2\pi \times a \times k_g = \pi$. And hence $area(D) = 2\pi - \pi = \pi$.

Grading Note: 1. For part a, we assume you have the compatible choice of normals with your answer, and we don't care about the sign of the geodesic curvature. But a common mistakes is that $k_g = k\sin\theta$, where k is the curvature of the curve, not the Gaussian curvature of the surface! For the answer using $k = 1$ by the Gaussian curvature, at most 7 points was given.

2. For part b, correctly write down the Gauss-Bonnet for the domain D will receive at least 10 points; and fail to apply correct Gauss-Bonnet will receive less than 9 pts. Fully correct spherical computation will also receive 15 pts, but I did not see this answer.

5.[30 pts] In general, a torus parameterized by $\Sigma(u,v) = ((R_1+R_2\cos v)\cos u, (R_1+R_2\cos v)\sin u, R_2\sin v)$ has $\quad I = (R_1+R_2\cos v)^2du^2 + R_2^2dv^2, \quad II = \pm[\cos v(R_1+R_2\cos v)du^2 + R_2dv^2].$



In the following questions, let S be the torus with $R_1 = 10, R_2 = 1$, i.e.

$$S = \sigma(u,v) = ((10+\cos v)\cos u, (10+\cos v)\sin u, \sin v), u \in [0,2\pi], v \in [0,2\pi];$$

and choose the **outward pointing normal vector**.

a.[21 pts] Find the principal curvatures of S at $A.(11,0,0) \quad B.(10,0,1) \quad C.(10.5,0,\frac{\sqrt{3}}{2})$. [**Hint:** The geometric approach might be faster, given the fact that the u-curves and v-curves are the principal directions].

b.[6 pts] Find the general formula of principal curvatures of S at the point $\sigma(u,v)$.

c.[3 pts] From part b), what is the Gaussian curvature and mean curvature at the point $\sigma(u,v)$?

Solution: a) We denote the u-direction by $\vec{t_1}$ and the v-direction by $\vec{t_2}$, and the corresponding principal curvatures k_1, k_2. The horizontal circle at $\Sigma(u,v)$ has curvature $\frac{1}{R_1+R_2\cos v}$, the vertical circles at $\Sigma(u,v)$ always have curvature $\frac{1}{R_2}$. And the angles between normal vectors of the curve and the surface: for the horizontal circles, the angle is π; and the vertical circle, the angle is $\pi - v$, since the surface normal is pointing outwards. So in general,

$$k_2 = \frac{1}{R_2}\cos\pi = -\frac{1}{R_2};$$

and

$$k_1 = \frac{1}{R_1+R_2\cos v}\cos(\pi - v) = \frac{-\cos v}{R_1+R_2\cos v}.$$

Now we return to the special case $\sigma(u,v)$. We have $A = \sigma(0,0)$, $k_2 = -1, k_1 = -1/11$; $B = \sigma(0,\frac{\pi}{2})$, $k_2 = -1, k_1 = 0$; $C = \sigma(0,\pi/3)$, $k_2 = -1, k_1 = -1/21$.

b) Plug in $R_1 = 10, R_2 = 1$, $k_2 = -1, k_1 = \frac{-\cos v}{10+\cos v}$.

c)The Gaussian curvature is $K = \frac{\cos v}{10+\cos v}$. The mean curvature is $H = \frac{-1}{2}(1+\frac{\cos v}{10+\cos v})$.

Note 1: As we already did in class and in the sample exam, on the surface of revolution we know the principal directions are the Meridian and Parallels (the u,v-curves). For details see the sample final.

Grading Note: 1.We emphasized that sign of k_1, k_2 is determined by the choice of surface normal vector. Here we choose the outward normal, which made them being negative. Not dealing with sign will be 2 pts off.

2.Other methods are also fine, either start with the general Gaussian and mean curvature or solving the eigenvalue of the matrix of the Weigarten map.

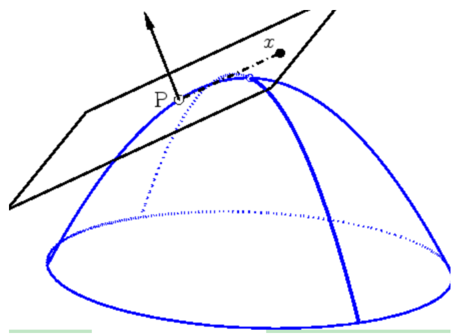
3. Numerical mistakes count at most 2 pts each, no repeated penalty.

6.[30 pts]Let S be a closed compact orientable surface in $\mathbb{R}^3$. $K(p)$ denotes the Gaussian curvature at the point $p \in S$. S^2 denotes the unit sphere centered at the origin.

a.[15 pts] Show that the Gauss map $N : S \to S^2$ is surjective, where we are choosing $\vec{n}$ to be the exterior unit normal vector.

b.[8 pts] Show that the Gaussian curvature positive part of the surface S has total curvature at least 4π. Namely, denote $K_+(p) = max\{0; K(p)\}, \forall p \in S$; and show that $\int_S K_+ dA \geq 4\pi$. [**Hint:** If you have no idea about this one, try to start with thinking about the torus in Q5].

c.[7 pts] Let S be a closed compact orientable surface as above, assume $K(p) \geq 1$ for all $p \in S$ and $area(S) \geq 4\pi$. Show that S must be isometric to the unit sphere S^2.



Solution: a(Same question on the sample) Choose any unit vector $\vec{v} \in S^2$. Then consider the plane given by equation $\vec{x} \cdot \vec{v} = A$, for any real number A. (Note in the picture, $\vec{x}$ is not the dashed line, and dashed line means projection of $\vec{x}$ onto the $\vec{v}$ direction.) For A very large, the plane does not intersect the given surface S (which is blue in the picture). Then we continuously change A smaller, until the first time the plane intersects the surface S. Denote the intersection point by P, and $N(P) = \vec{v} \in S^2$. Hence the subjectivity holds.

Grading Note: Not being able to clearly show how to find the preimage of $\vec{v}$ will receive at most 10 pts.

Let S_+ be the surface with positive Gaussian part. Since by part a), the point we found lies in one side of the tangent plane, it must be on S_+. Then the **key point** is that Gaussian curvature K is by definition the determinant of the differential of Gauss map, i.e.

$$K = det(dN).$$

Only from here can we connect the surface total curvature with the sphere: $\int_S K_+ dA = \int_{S_+} K dA = \int_{S_+} det(dN) dA = \int_{N(S_+)} dN(A)$. Note that $N(A)$ is the area element on S^2, and the right hand side is no less than 4π by the surjectivity of the Gauss map from part a). Hence we finished the proof.

Grading Note: Without the key point that $K = det(dN)$, the proof cannot be complete, and will receive at most 6 pts. Such incomplete proofs include "gluing together the $K > 0$ part," direct comparison of integrals without $K = det(dN)$, or think about the inverse of Gauss map. Also note that the Gauss map of the S_+ part could also be non-injective, just think about a high genus surface.

c) We have

$$4\pi \leq area(S) \leq \int_S K dA = 2\pi\chi(S) \leq 4\pi.$$

In the last inequality, we use the classification of compact orientable surfaces(strictly speaking here we emphasize that the surface is connected, otherwise it's possible to have many spherical components). Then from the above pinch of 4π and the fact that K is a smooth function, we know that firstly $\chi(S)$ has to be 2(which means homeomorphic to a sphere), furthermore $K \equiv 1$. Then by classification of constant curvature surfaces we know S has to be isometric to a unit sphere.

Grading Note: I basically saw 3 solutions, where almost everyone did not use the surface is connected (which is essential):

1. Having a similar argument as above (maybe missing connectedness), we give full credits, since in the class by saying a surface we often suppose it's connected.
2. If people ended up assuming connectedness and only proved S is homeomorphic to a sphere, this is not enough. Since being isometric is much stronger than just being homeomorphic. For this case we gave 5 points.
3. If people only mentioned classification of surfaces, then will receive 3 points.