Math 433 W 2019 - Solutions to HW for selected problem on regular curves

Note that Exam 1 is on Feb 20, 11:00(sharp)-11:50, which covers 1.1 through 2.4 in do Carmo's book.

As requested by some people we post selected problem (mostly from week 3) on regular curves, and is aimed to get everyone prepared for the Exam 1. Some previous grading materials are used, which is hand written and I hope that will be clear.

There will be 4 problems in Exam 1, where 2 on curves and 2 on surfaces, you are allowed to use at most one 3 inch by 5 inch note card, two sided, with anything you want written on the card.

1.. Many question is about computation of curvature and torsion of a curve (not necessarily arc-length parametrized). For example, 3,4: Q1, Q2, Q4.

For Q1, in general, if a helix is parametrized as $(Acost, Asint, Bt)$, then its curvature is $\frac{|A|}{A^2+B^2}$, note that curvature is non-negative; and its torsion is $\frac{B}{A^2+B^2}$, note that torsion could be negative. Both can be computed by the formula of Q1-5-12.

2.. In the lectures we mentioned part b), c) of Q1-5-12. on page 26 of 2nd edition of the book. This question can be applied to solve 3,4: Q1, Q2, Q4.

1-5-12: Here $\alpha : I \to \mathbb{R}^3$ is a regular parametrized curve, not necessarily by arc length. And we have $\beta : J \to \mathbb{R}^3$ be its arclength repara by $s = s(t)$.

b)Curvature of $\alpha(t)$ is $k(t) = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$. (we proved this in class and below we still post it here.)

c) Torsion of $\alpha(t)$ is $\tau(t) = -\frac{(\alpha' \wedge \alpha'') \cdot \alpha'''}{|\alpha' \wedge \alpha''|^2}$.

(b). proof: $\alpha(t)$, $\beta(s)$ as above, we take derivative

$$\begin{aligned}
\alpha'' &= \frac{d}{dt}\alpha' \\
&= \frac{d}{dt}\left(\beta \frac{ds}{dt}\right) \\
&= \frac{d}{dt}(\beta')\frac{ds}{dt} + \beta' \frac{d^2 s}{dt^2} \\
&= \beta''(\frac{ds}{dt})^2 + \beta' \frac{d^2 s}{dt^2} \qquad \text{by applying the chain rule} \\
&= k(s)n(s)(\frac{ds}{dt})^2 + \beta' \frac{d^2 s}{dt^2} \qquad \text{by Frenet formulas}
\end{aligned}$$

Then by applying the linearity and that $\beta' \wedge \beta' = 0$

$$\begin{aligned}
\alpha' \wedge \alpha'' &= \left(\beta'(s)\frac{ds}{dt}\right) \wedge \left(k(s)n(s)(\frac{ds}{dt})^2 + \beta' \frac{d^2 s}{dt^2}\right) \\
&= (\frac{ds}{dt})^3 k(s) \left(\beta' \wedge n(s)\right) \\
&= |\alpha'|^3 k(s) \left(\beta' \wedge n(s)\right)
\end{aligned}$$

by taking norms and rearranging we have $|k(s)| = \frac{|\alpha' \wedge \alpha''|}{|\alpha'|^3}$. Then since $k(s) = k(t)$ we have shown the formula. $\square$

(c).

Proof. From part (b) we have $\alpha'' = \beta''(\frac{ds}{dt})^2 + \beta'\frac{d^2s}{dt^2}$. Differentiating by $\frac{d}{dt}$ gets

$$\begin{aligned}
\alpha''' &= \beta'''(\frac{ds}{dt})^3 + 3\beta''(\frac{d^2s}{dt^2})(\frac{ds}{dt}) + \beta'\frac{d^3s}{dt^3} \\
&= (\frac{ds}{dt})^3\left[-k\beta' - \tau\vec{b}\right] + 3\beta''(\frac{d^2s}{dt^2})(\frac{ds}{dt}) + \beta'\frac{d^3s}{dt^3} \quad \text{By Frenet eqns} \\
&= \left(\frac{d^3s}{dt^3} - k(\frac{ds}{dt})^3\right)\beta' + 3(\frac{d^2s}{dt^2})(\frac{ds}{dt})\beta'' - (\frac{ds}{dt})^3\tau\vec{b}
\end{aligned}$$

By a formula on page 12 $(\alpha'\wedge\alpha'')\cdot\alpha''' = \det(\alpha', \alpha'', \alpha''') = \begin{vmatrix} \frac{ds}{dt}, & 0, & 0 \\ \frac{ds}{dt}, & (\frac{ds}{dt})^2, & 0 \\ \frac{d^3s}{dt^3} - k(\frac{ds}{dt})^3, & 3(\frac{d^2s}{dt^2})(\frac{ds}{dt}), & \frac{ds}{dt})^3\tau \end{vmatrix} =$ $-(\frac{ds}{dt})^6\tau$ Next $|\alpha' \wedge \alpha''| = (\frac{ds}{dt})^3$ by directly computing a cross product. Putting these two together we get the formula $\tau = -\frac{(\alpha'\wedge\alpha'')\cdot\alpha'''}{|\alpha'\wedge\alpha''|^2}$ as wanted. □

3. 3,4, Q3 b):

We mentioned this in class. Some people solved this solution by saying that we allowed the curve to be piecewise smooth and then it could be the graph of $|x|$, which is not a straight line.

However, by a "regular parametrized curve", we usually mean a smooth parametrization. The point of this question is that even through the trace is not smooth, we can still find a **smooth parametrization**, by using $(t^2, t|t|)$ for example, or any power function $(t^n \cdot t, t^n \cdot |t|)$ $t^n, n > 0$, such that the parametrization is smooth.