Math 433 W 2019

- Sample Final with solutions. On Apr 22 Monday we'll give a review.

- The final time will be Apr 29, 4:00(sharp)-6:00 pm. EH4088 -

Name:

There are 6 questions in the final, totally counting 150 points. You are allowed to use at most one 3-inch by 5-inch two-sided note card.

Remember that your work is graded on the quality of your writing and explanation as well as the validity of the mathematics. Partial credits will be given for partially correct arguments. The format of the final will be similar to this one. Questions will be different, but will cover similar contents and will be at a similar level of difficulty.

1.[30 pts] Calculate the Frenet frame $(\vec{t}; \vec{n}; \vec{b})$ **and the curvature** k**, torsion** τ **for**

$$\gamma(t) = (\cos t; \sin t; t), \quad t \in (-1; 1).$$

Solution: Parametrization by arc length: $\alpha(s) = (\cos(s/\sqrt{2}); \sin(s/\sqrt{2}); (s/\sqrt{2}))$.

$$\vec{t} = \alpha'(s) = \frac{1}{\sqrt{2}}(-\sin(s/\sqrt{2}); \cos(s/\sqrt{2}); 1).$$

$$\alpha''(s) = \frac{1}{2}(-\cos(s/\sqrt{2}); \sin(s/\sqrt{2}); 0).$$

Then $k = \frac{1}{2}$ **and** $\vec{n} = (-\cos(s/\sqrt{2}); \sin(s/\sqrt{2}); 0)$.

$$\vec{b} = \vec{t} \wedge \vec{n} = \frac{1}{\sqrt{2}}(\sin(s/\sqrt{2}); -\cos(s/\sqrt{2}); 1).$$

Then $\vec{b}' = \frac{1}{\sqrt{2}}(-\cos(s/\sqrt{2}); -\sin(s/\sqrt{2}); 0)$; **and because** $\vec{b}' = \frac{1}{\sqrt{2}}\vec{n}$**, we know** $\tau = \frac{1}{\sqrt{2}}$.

Note: The only thing we want to note here is that to calculate the Frenet frame, we need to start with the arc-length parametrization.

2.True or False with brief explantation [5pts× 3]

a. For any given smooth functions $E; F; G$ **and** $L; M; N$**, there exists a regular surface** S **with** $I_S = Ex^2 + 2Fxy + Gy^2$ **and** $II_S = Lx^2 + 2Mxy + Ny^2$.

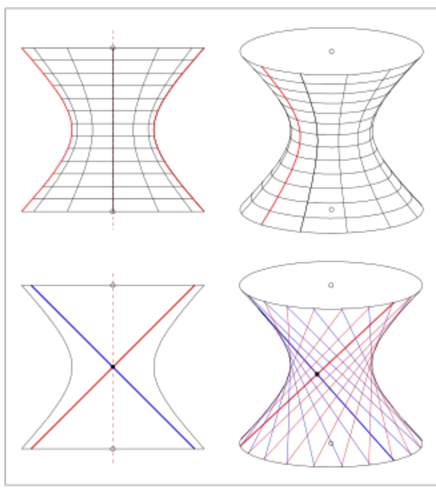
Solution: False. Fundamental theorem of surfaces, there need to be some comparability equations (Gauss equations and Codazzi equations) on E, F, G **and** L, M, N**.**

b. Let L **be a straight line in a surface** S **in** $\mathbb{R}^3$**. Then, for any** $p \in L$ **the direction given by** L **is principal direction.**

Solution: False. Consider the hyperboloid H **with equation**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Geometrically one can tell everywhere the surface has $K < 0$**. This means there is one positive principal curvature and one negative curvature. However, we can find lines on** H **by letting** $x = a$**. Those lines has curvature 0, and hence can not give principal directions.**



Note: This means that the hyperboloid H **is a ruled surface(which is the union of straight lines).**

c. Let S **be a (homeomorphic) disk in** $\mathbb{R}^3$ **with Gaussian curvature** $K \leq 0$ **everywhere. Then, any two different geodesics starting at the same point cannot meet again in** S.

Solution: True. Q9 in HW.

3. [20pts] **We define a dilation map** $F : \mathbb{R}^3 \to \mathbb{R}^3$ **to be** $F(x, y, z) = (2x, 2y, 2z)$.

a.Let $\gamma(s)$ **be a regular curve in** $\mathbb{R}^3$, **and let** $\bar{\gamma}(s) = F(\gamma(s))$. **Compute** $\bar{k}(s)$ **and** $\bar{\tau}(s)$ **of** $\bar{\gamma}(s)$ **in terms of** $k(s)$ **and** $\tau(s)$ **of** $\gamma(s)$.

b. Let S **be a regular surface, and let** $\bar{S} = F(S)$. **Compute Gaussian curvature** $\bar{K}(p)$ **of** $\bar{S}$ **in terms of** $K(p)$ **of** S.

Solution:

a)$\bar{\gamma}(s) = F(\gamma(s)) = 2\gamma(s)$. **Then the arc-length parametrization of** $\bar{\gamma}(s)$ **should be** $2\gamma(s/2)$, **because** $|\bar{\gamma}(s)'| = 2|\gamma(s)'| = 2$.

Now we know $\bar{\gamma}(s)' = \gamma'(s/2)$ **and** $\bar{\gamma}(s)'' = \frac{1}{2}\gamma'(s/2)$. **Then we know** $\bar{k}(s) = \frac{1}{2}k(s/2)$ **and** $\bar{\tau}(s) = \frac{1}{2}\tau(s/2)$.

b) Choose two lines of curvature at any point p on S, **parameterizing them by arc-length. Their curvature are the principal curvatures** k_1, k_2. **Now choose the point** $F(p)$ **on surface** $\bar{S}$. **By part a, the image of the pair of curves will have curvatures** $k_1/2, k_2/2$ **at** $F(p)$. **Now we know that the Gaussian curvature at** $F(p)$ **(i.e.** $\bar{K}(F(p))$**) is** $1/4$ **of** $K(p)$.

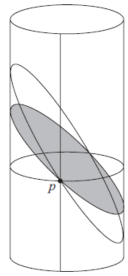
Note: The point of this question is reparameterization. Indeed there's two approaches to do an arc-length reparametrazation of the curve.

One is the way we do in part a, replace s **by** $s/|\gamma'|$ **in the function of the curve. Our intuition for this is to make the curve travel slower.**

The other way is to change the variable to be $s \times |\gamma'|$ **and use the original function, as we do in the note of Q4. Our intuition for this is to make the variable travel faster.**

For different situation we apply the appropriate reparameterization.

4.[25 pts] **Intersect the cylinder $x^2 + y^2 = 1$ with a plane passing through the x axis and making an angle $|\theta| \neq \pi/2$ with the xy plane. Show that the intersecting curve is an ellipse C. Compute the geodesic curvature of C in the cylinder at the points where C meets its axes.**



Solution: Firstly we prove the intersection is an ellipse: The equation of plane P can be written as $z = ay$. On plane P we set $\bar{x} = x$, $\bar{y} = \sqrt{1+a^2}y$. Then on plane P we have an orthogonal coordinate $(\bar{x}, \bar{y})$, and the curve C has equation $\bar{x}^2 + \frac{\bar{y}^2}{1+a^2} = 1$. Hence it's an ellipse.

Next, we compute the geodesic curvatures at the 4 intersection points. These will four points on this picture: a top one, two middle ones and a bottom one.

The geodesic curvature of the top and the bottom ones is obtained by computing the curvature of the ellipse at such a point and multiplying it by the sine of the angle between the curve normal and the surface normal (choose the inward orientation), which is the same as θ. Thus, the answer will be $k \sin\theta$, where θ is the curvature of the ellipse at a point where it meets its long axis, as computed as following.

Parameterize the ellipse by $u = x$ and $v = y/cos\theta$. This is a parametrization of the plane P in which the uv distance will be the same as the distance in space. In this plane P, the ellipse will be given by the equation $u^2 + v^2\cos^2(\theta) = 1$, (note that this is the same as the equation we worked out as above). Let $b = \frac{1}{\cos(\theta)}$, and parameterize the ellipse $E(t)$ by $u = \cos t, v = b\sin t$, where $E(0)$ being the point p as in the picture.(Note this is not an arc-length parameter, and to do by arc-length reparametrization is indeed complicated). The best way is to use the curvature formula of a plane curve with general parametrization: $k = |\frac{u'v''-v'u''}{(u'^2+v'^2)^{3/2}}| = |\frac{-b\sin^2 t - b\cos^2 t}{(\sin^2 t + b^2\cos^2 t)^{3/2}}|$. When $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, we easily have $k = b = 1/\cos\theta$, and hence $k_g = \frac{\sin\theta}{\cos\theta}$.

The geodesic curvature at the middle ones will be zero, because the normal to the ellipse at these points will point to its center, which is on the central axis of the cylinder and therefore parallel to the normal of the cylinder.

Note 1: One can also use the following solution to solve the curvature at $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$. There's indeed less computation than expected: let $\vec{T} = \gamma'(t) = (-\sin t, b\cos t)$, reparametrize by arc-length we have $s = \int |\vec{T}|dt$, and $\frac{ds}{dt} = |(-\sin t, b\cos t)|$, and $k = |\frac{d\vec{T}}{ds}|$. We then compute $\frac{d\vec{T}}{ds}$ from the relation

$$\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt}.$$

At $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, we have $\frac{ds}{dt} = |(-\sin(\pi/2), b\cos t)| = 1$. This means

$$|\frac{d\vec{T}}{ds}| = |\frac{d\vec{T}}{dt}| = |(0, -b)| = b.$$

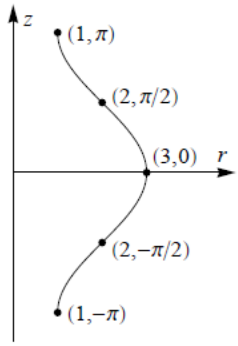
Hence at $t = \frac{\pi}{2}$ or $t = \frac{3\pi}{2}$, $k_g = k\sin\theta = \frac{\sin\theta}{\cos\theta}$.

5.[30 pts] Let S be the surface $r = 2 + \cos z$ for $-\pi < z < \pi$ (in cylindrical coordinates), oriented with normal vectors pointing outwards. It is a surface of revolution, by rotating the curve $y = 2 + \cos z$ in yz plane along z-axis.

a. Find the principal curvatures of S at the point $(3, 0, 0)$.

b. Find the principal curvatures of S at the point $(2, 0, \pi/2)$.

c. For what values of z in the range $-\pi < z < \pi$ is the Gaussian curvature of S positive?(write your answer with a brief explanation will be fine.)



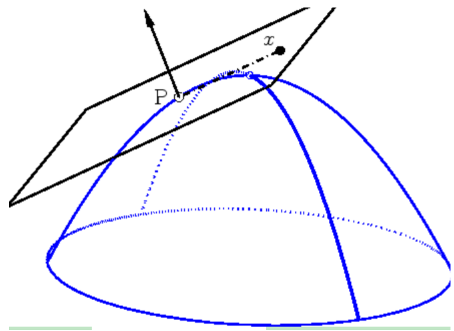
Solution: a) The curve $r = 2 + \cos z$ on the rz-plane is shown above. This curve will be a geodesic(HW Q1) on the resulting surface, and its curvature at $(3; 0; 0)$ is 1. The horizontal circle at $z = 0$ is also a geodesic, with a curvature of $1/3$. Both curves are bending away from the normal vector (note we choose the outward normal on the surface), so the principal curvatures at $(3; 0; 0)$ are -1 and $-1/3$.
b)Again, the curve shown above is a geodesic, and its curvature at $(2; 0; \pi/2)$ is 0. The horizontal circle at $z = \pi/2$ has a curvature of $1/2$, but it is not a geodesic (HW Q1). Its principal normal vector is $(-1; 0; 0)$, and the normal vector to the surface is $(1; 0; 1)/\sqrt{2}$, by Euler's theorem $k_n = -\frac{1}{2\sqrt{2}}$. Hence the principle curvatures are 0 and $-\frac{1}{2\sqrt{2}}$.
c)The Gaussian curvature will be positive for $\frac{-\pi}{2} < z < \frac{\pi}{2}$.
A brief explanation is that the horizontal circles always give negative principal curvature. And the above curve only have negative curvature for $\frac{-\pi}{2} < z < \frac{\pi}{2}$.

6.[30 pts]

a. Let S be a closed compact orientable surface of $\mathbb{R}^3$. K denotes the Gaussian curvature of S. S^2 denotes the unit sphere centered at the origin. Show that the Gauss map $N : S \to S^2$ is surjective, where we are choosing $\vec{n}$ to be the exterior unit normal vector.

b. Show that if the closed compact orientable surface has everywhere positive Gaussian curvature, then it must be homeomorphic to S^2.

c. Let S be any surface. Is the Gauss map $N : S \to S^2$ necessarily surjective? If no, give an example. If yes, explain why.



Solution: **a)Choose any unit vector $\vec{v} \in S^2$. Then consider the plane given by equation $\vec{x} \cdot \vec{v} = A$, for any real number A. (Note in the picture, $\vec{x}$ is not the dashed line, and dashed line means projection of $\vec{x}$ onto the $\vec{v}$ direction.) For A very large, the plane does not intersect the given surface S (which is blue in the picture). Then we continuously change A smaller, until the first time the plane intersects the surface S. Denote the intersection point by P, and $N(P) = \vec{v} \in S^2$. Hence the subjectivity holds.**

b)classification of compact orientable surfaces and global Gauss-Bonnet. See textbook p.280.

c)No. Consider the plane, image of Gauss map is a point.

Note: **The only thing we want to note here is that part a) of this question is related to Q8 in the HW, in the sense that those point $P \in S$ we found in part a) must have non-negative Gaussian curvature. Indeed, on a compact surface the Gaussian curvature non-negative part under the Gauss map will cover the whole sphere S^2.**