Math 433 W 2019 - Solution of HW for curvatures

Not for turn in, but for preparation of Exam 2.

The exam policy is the same as Exam 1, one $5' \times 3'$ note card is allowed.

Lesson plan, reading assignments and homework:

Week 9-10: Sections 4-3,2-4 in do Carmo's book, 1st step to global geometry.

This HW contains 5 problems from do Carmo's book, and is aimed to preparing for the Exam 2.

We strongly recommend everybody to try each problem by oneself before the solution is posted.

1.. What is the Gaussian curvature of a surface of revolution?

Hint: Use the "good" parametrization we gave in class.

Proof:

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u)).$$

We again assume $f_u^2 + g_u^2 = 1$ *and* $f > 0$.

$$I = du^2 + f^2 dv^2, II = (f_u g_{uu} - f_{uu} g_u) du^2 + f g_u dv^2.$$

Hence

$$K = \frac{(f_u g_{uu} - f_{uu} g_u) f g_u}{f^2}.$$

Taking derivative on $f_u^2 + g_u^2 = 1$, *we have*

$$f_u f_{uu} + g_u g_{uu} = 0.$$

So

$$(f_u g_{uu} - f_{uu} g_u) g_u = -f_{uu}(f_u^2 + g_u^2) = -f_{uu},$$

and

$$K = -\frac{f_{uu} f}{f^2} = -\frac{f_{uu}}{f}.$$

Especially, for a unit sphere $u = \theta, v = \phi, f(\theta) = \cos\theta, g(\theta) = \sin\theta$. *We thus have* $K = 1$.

Note: If you are far ahead, after Exam 2 we'll prove a general formula for Gaussian curvature:

By formulas on page 239 in do Carmo, $\Gamma_{11}^1 = \frac{E_u}{2E}, \Gamma_{11}^2 = \frac{E_v}{2G}, \Gamma_{12}^1 = \frac{E_v}{2E}, \Gamma_{12}^2 = \frac{G_u}{2G}, \Gamma_{22}^1 = \frac{G_u}{2E}, \Gamma_{22}^2 = \frac{G_v}{2G}$.

Hence by Gauss equation

$$EK = -(\frac{E_v}{2G})_v - (\frac{G_u}{2G})_u + \frac{E_u G_u}{4EG} - \frac{E_v G_v}{4G^2} + \frac{E_v^2}{4EG} - \frac{G_u^2}{4G^2}.$$

Take derivative and simplify,

$$\begin{aligned} -2K\sqrt{EG} &= \frac{E_{vv} + G_{uu}}{\sqrt{EG}} - \frac{E_v(EG_v + E_v G)}{2\sqrt{EG}^3} - \frac{G_u(GE_u + EG_u)}{2\sqrt{EG}^3} \\ &= \frac{E_{vv}}{\sqrt{EG}} - \frac{E_v(EG)_v}{2\sqrt{EG}^3} + \frac{G_{uu}}{\sqrt{EG}} - \frac{G_u(EG)_u}{2\sqrt{EG}^3} \\ &= \frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}. \end{aligned}$$

For a surface of revolution

$$\sigma(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

where the base curve is arc-length parameterized, i.e. $f_v^2 + g_v^2 = 1$, we already know that $E = f^2; F = 0; G = f_v^2 + g_v^2 = 1$. By the above formula, $K = -\frac{f_{vv}}{f}$.

2. Show that the mean curvature H at $p \in S$ is given by

$$H = \frac{1}{\pi}\int_0^\pi \kappa_n(\theta)d\theta,$$

where $\kappa_n(\theta)$ is the normal curvature at p along a direction making an angle θ with a fixed direction.

Proof:

By Euler's theorem (introduced on the 1st class after the break),

$$k_n = k_1\cos^2\theta + k_2\sin^2\theta,$$

where θ is the oriented angle from the tangent direction to t_1(the principal direction corresponding to k_1.)

Then $k_n(\theta) = k_1\cos^2(\theta+\theta_0) + k_2\sin^2(\theta+\theta_0)$.

Hence

$\frac{1}{\pi}\int_0^\pi k_n(\theta)d\theta = \frac{1}{\pi}k_1\int_0^\pi \cos^2(\theta+\theta_0)d\theta + \frac{1}{\pi}k_2\int_0^\pi \sin^2(\theta+\theta_0)d\theta = \frac{1}{2}(k_1+k_2)H.$

3. If the surface S_1 and S_2 intersect along a regular curve C, then the curvature k of C at p is given by

$$k^2 \sin^2 \theta = \lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta,$$

where λ_1 and λ_2 are the normal curvatures at p, along the tangent line to C, of S_1 and S_2, respectively, and θ is the angle made up by the normal vectors of S_1 and S_2 at p.

Solution: Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be unit normal vectors of S_1 and S_2 respectively, and $\mathbf{n}$ be the principal normal of C.

So $\lambda_1 = k\mathbf{N}_1 \cdot \mathbf{n}$, $\lambda_2 = k\mathbf{N}_2 \cdot \mathbf{n}$. Hence

$$k||(\mathbf{n} \cdot \mathbf{N}_1)\mathbf{N}_2 - (\mathbf{n} \cdot \mathbf{N}_2)\mathbf{N}_1|| = ||\lambda_1\mathbf{N}_2 - \lambda_2\mathbf{N}_1|| = \sqrt{\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2 \cos\theta}$$

because $\mathbf{N}_1 \cdot \mathbf{N}_2 = \cos\theta$.

On the other hand,

$$||(\mathbf{n} \cdot \mathbf{N}_1)\mathbf{N}_2 - (\mathbf{n} \cdot \mathbf{N}_2)\mathbf{N}_1|| = ||\mathbf{n} \times (\mathbf{N}_1 \times \mathbf{N}_2)|| = ||\mathbf{N}_1 \times \mathbf{N}_2|| = |\sin\theta|.$$

This completes the equality.

Note: The last second equality $||\vec{n}\wedge(N_1\wedge N_2)|| = ||N_1\wedge N_2||$ comes from the fact that $\vec{n}$ is orthogonal to $N_1 \wedge N_2$. This is because both N_1, N_2 are orthogonal to the tangent vector of the curve, hence $N_1 \wedge N_2$ is the tangent direction of the curve. And hence $\vec{n}$ is orthogonal to $N_1 \wedge N_2$.

Indeed, we don't plan to put a question like 2 or 3 in the exam 2, but we will have computation of normal curvatures, or questions on principle curvatures (for example, to determine whether a point is elliptic, hyperbolic, parabolic, or planer, umbilical etc.) Hence these two questions are still helpful in the sense that they reminds us about those concepts.

4. This is indeed not a examinable problem, but can be seen as a continuation of Q4 in the previous homework(was around do Carmo page 230):

It's a well-known but difficult theorem that any C^2 (having continuous 2nd order derivative) surface in $\mathbb{R}^3$ with **parameter domain the unit disk** (just means the unit disk in the (u, v) plane of the local chart) can be put into isothermal coordinates. Here isothermal coordinates means a parametrization with $I = \lambda(u, v)du^2 + \lambda(u, v)dv^2$, where $\lambda(u, v)$ is a everywhere positive smooth function in the unit disk. Note that by reparameterizing, one can replace the above "unit disk" by any simply connected open subset in the local chart. Or simply put, **locally there's always a isothermal coordinate on a smooth surface**.

An immediate corollary is that **any two surfaces are locally conformal.** Then it is not surprising at all that we can do conformal diffeomorphism to make a planar world map.

The proof is long and beyond the scope of the class, go here for the proof

`http://www.math.titech.ac.jp/~kotaro/class/2016/geom-f/lecture-03.pdf`

And here for some interesting application in computer vision

`http://www3.cs.stonybrook.edu/~gu/talks/Barrett_2010/ConformalMapping.pdf`

We shall high light several examples on surface of revolution, which is helpful for Exam 2, as well as future lectures on Gauss-Bonnet.

- Sphere parameterized by $\sigma(u,v) = (R\cos u\cos v, R\cos u\cos v, R\sin v)$.

$$I = R^2du^2 + (R\cos u)^2dv^2, II = Rdu^2 + R\cos^2 udv^2.$$

  Gaussian curvature at each point is $K = \frac{1}{R^2}$; mean curvature at each point is $H = \frac{1}{R}$.

- Cylinder parameterized by $\sigma(u,v) = (R\cos u, R\sin u, v)$.

$$I = R^2du^2 + dv^2, II = -Rdu^2.$$

  Gaussian curvature at each point is $K = 0$; mean curvature at each point is $H = -\frac{1}{R}$.

- Torus parameterized by $\sigma(u,v) = ((a + R\cos v)\cos u, (a + R\cos v)\sin u, R\sin v)$.

$$I = (a + R\cos v)^2du^2 + R^2dv^2, II = \cos v(a + R\cos v)^2du^2 + Rdv^2.$$

  Gaussian curvature at each point is $K = \frac{\cos v}{R(a+R\cos v)}$; mean curvature at each point is $H = \frac{a+2R\cos v)}{2R(a+R\cos v)}$.