Math 433 W 2019 - Week 12-13,

HW 5 and 6 merged, due on Apr 19, Friday

Lesson plan, reading assignments and homework:

Week 12-13: Sections 4-4, 4-5, 4-6, 4-7 in do Carmo's book, geodesics and Gauss-Bonnet Theorem. There are 10 questions totally counting 200 points, since they are the union of two standard homework sets.

You'll also find Hitchin's notes Page 65-75 very useful. Especially Page 74 the geodesic equations. These two weeks is the 1st step in Riemannian geometry. Almost everything we talk in class has generalizations in any higher-dimensional Riemannian manifolds.

1.[20pts] For $\{0 < u < 2\pi, a < v < b\} \subset \mathbb{R}^2$, the map

$$\sigma(u, v) = (f(v)\cos u, f(v)\sin u, g(v)) \subset \mathbb{R}^3$$

is a surface of revolution. Assume the profile curve on the xz plane $\gamma(v) = (f(v), g(v)), \quad a < v < b, \quad f(v) > 0$, is arc-length parameterized, i.e. $f_v^2 + g_v^2 = 1$. We have

$$I = f(v)^2 du^2 + dv^2.$$

a)show that the geodesic equation in this case is

$$\ddot{v} = f(v)\frac{df}{dv}(\dot{u})^2,$$

and

$$\frac{d}{dt}(f^2(v)\dot{u}) = 0.$$

b)show that the meridian $u \equiv C$ is a geodesic.

c)show that a parallel $v \equiv C$ is a geodesic iff $f_v(C) = 0$.

d)In the torus $\sigma(u, v) = ((100 + \cos v)\cos u, (100 + \cos v)\sin u, \sin v) \subset \mathbb{R}^3$, there's a circle $z = 1$ on top. Is it a geodesic?

Remark: Besides doing the calculation of geodesic equations, we can try to use the tangent cone idea to visualize whether or not a parallel is a geodesic.

Here's an online demo of parallel transport on the sphere, and your browser needs to support Java: http://torus.math.uiuc.edu/jms/java/dragsphere/

As we mentioned in the March 24th class, if we parallel transport a vector along a non-geodesic closed loop in S^2(indeed we can do in any surface), the vector does not get back to itself. However, if we parallel transport a vector along a geodesic loop, the vector will come back to itself.

2. [20pts] Compute the geodesic curvature of the upper parallel (circle $z = 1$) of the torus $\sigma(u, v) = ((100 + \cos v)\cos u, (100 + \cos v)\sin u, \sin v) \subset \mathbb{R}^3$.

Before we do 3, here's a simple definition we now introduce (see page 147 in do Carmo):
If a regular connected curve C on S satisfies this property: for all $p \in C$, the tangent line of C is a principal direction at p; then C is said to be a **line of curvature** of S.

3. a) Show that if a geodesic whose curvature is nowhere 0 is also a line of curvature, then it is a plane curve. (Hint: assume arc-length, use the local frame.)
b) Give an example of a line of curvature that is not a geodesic.

4. [10pts] Let $\vec{v}, \vec{w}$ be tangent vector fields along a curve $\gamma : (a, b) \to S$. Show that

$$\frac{d}{dt} < v(\vec{t}), w(\vec{t}) >=< \nabla_\gamma v(t), w(\vec{t}) > + < v(\vec{t}), \nabla_\gamma w(t) > .$$

[Hint: use the definition of the covariant derivative and the computation should be very easy.]

5. **a.** Show that if σ is an isothermal parametrization, that is, $E = G = \lambda(u, v)$ and $F = 0$, then the Gaussian curvature

$$K = -\frac{1}{2\lambda}\Delta(\ln \lambda),$$

where $\Delta\phi$ denotes the Laplacian $\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial v^2}$ of the function ϕ.

b. Calculate the Gaussian curvature of the surface (upper half-plane model) with first fundamental form

$$\frac{dv^2 + du^2}{u^2}.$$

6. [10pts] In geodesic normal coordinates $I = du^2 + G(u, v)dv^2$. Show that

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

[Hint:] Start with $K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$, if you get stuck, compute $\frac{\partial\sqrt{G}}{\partial u}$ and you'll see the pattern.

7. [10pts] Geodesics are solutions to the 1-dimensional variational problem. Minimal surfaces are the 2-dimensional analog of geodesics because it's the local area minimizer. A surface in $\mathbb{R}^3$ is called a minimal surface if it has zero mean curvature at every point.

a)Show that a minimal surface must have negative Gaussian curvature everywhere. (Consider principal curvatures.)

b)Show that there's no compact minimal surface embedded in $\mathbb{R}^3$. (Recall a fact that there must be a point on the compact surface in $\mathbb{R}^3$ with positive Gaussian curvature.)

Here's some illustration of Minimal Surfaces. We actually see them a lot in daily life, for example, the shape of a soap film.
http://torus.math.uiuc.edu/jms/Images/almgren/

8.(do Carmo 4-5, Q1:)[30pts]

Let $S \subset R^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

[Hint:] Use global Gauss-Bonnet, and you'll again find the following fact useful: there must be a point on a compact surface in $\mathbb{R}^3$ with positive Gaussian curvature.

9. [30pts] Let $S \subset \mathbb{R}^3$ be a surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 on S which start at a point p cannot meet again at a point q in such a way that together they bound a region S' on S which is homeomorphic to a disk.
[Hint:] Local Gauss-Bonnet will help.

10. [30pts] Let $S \subset \mathbb{R}^3$ be a surface homeomorphic to a cylinder and with Gaussian curvature $K < 0$. Show that S has at most one simple closed geodesic.
[Hint:] Q9 will help.

The rest questions are to do for your own, not for turn in. Solutions will also be posted on Friday, Apr 20th.

1.(Counterpart of do Carmo 4-5, Q1:)

If the surface $S \subset \mathbb{R}^3$ is not assumed to be compact, are there still always points with $K = 0, K > 0, K < 0$?

2.(do Carmo 4-5, Q2:)

Let T be a torus of revolution. Describe the image of the Gauss map of T and show, without using the Gauss-Bonnet theorem, that

$$\iint_T K\, d\sigma = 0.$$

Compute the Euler-Poincaré characteristic of T and check the above result with the Gauss-Bonnet theorem.

3.(do Carmo 4-5, Q4:)

Compute the Euler-Poincaré characteristic of

a. An ellipsoid.

b. The surface $S = \{(x, y, z) \in R^3; x^2 + y^{10} + z^6 = 1\}$.

4.(do Carmo 4-6, Q9:)

(*A Local Isoperimetric Inequality for Geodesic Circles.*) Let $p \in S$ and let $S_r(p)$ be a geodesic circle of center p and radius r. Let L be the arc length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + R,$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{R}{r^4} = 0.$$

Thus, if $K(p) > 0$ (or < 0) and r is small, $4\pi A - L^2 > 0$ (or < 0). (Compare the isoperimetric inequality of Sec. 1-7.)

5.(do Carmo 4-6, Q13:)(You need to know what is a group for this question, this is not required on the exam. But doing this definitely helps understanding parallel transport and Gauss-Bonnet.)

(*The Holonomy Group.*) Let S be a regular surface and $p \in S$. For each piecewise regular parametrized curve $\alpha: [0, l] \to S$ with $\alpha(0) = \alpha(l) = p$, let $P_\alpha: T_p(S) \to T_p(S)$ be the map which assigns to each $v \in T_p(S)$ its parallel transport along α back to p. By Prop. 1 of Sec. 4-4, P_α is a linear isometry of $T_p(S)$. If $\beta: [l, \bar{l}]$ is another piecewise regular parametrized curve with $\beta(l) = \beta(\bar{l}) = p$, define the curve $\beta \circ \alpha: [0, \bar{l}] \to S$ by running successively first α and then β; that is, $\beta \circ \alpha(s) = \alpha(s)$ if $s \in [0, l]$, and $\beta \circ \alpha(s) = \beta(s)$ if $s \in [l, \bar{l}]$.

a. Consider the set

$$H_p(S) = \{P_\alpha: T_p(S) \to T_p(S); \text{ all } \alpha \text{ joining } p \text{ to } p\},$$

where α is piecewise regular. Define in this set the operation $P_\beta \circ P_\alpha = P_{\beta \circ \alpha}$; that is, $P_\beta \circ P_\alpha$ is the usual composition of performing first P_α and then P_β. Prove that, with this operation, $H_p(S)$ is a group (actually, a subgroup of the group of linear isometries of $T_p(S)$). $H_p(S)$ is called the *holonomy group* of S at p.

b. Show that the holonomy group at any point of a surface homeomorphic to a disk with $K \equiv 0$ reduces to the identity.

c. Prove that if S is connected, the holonomy groups $H_p(S)$ and $H_q(S)$ at two arbitrary points $p, q \in S$ are isomorphic. Thus, we can talk about *the* (abstract) *holonomy group of a surface*.

d. Prove that the holonomy group of a sphere is isomorphic to the group of 2×2 rotation matrices (cf. Exercise 22, Sec. 4-4).

6.(do Carmo 4-4, Q3:)

Verify that the surfaces

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, \log u), \quad u > 0$$
$$\bar{\mathbf{x}}(u, v) = (u \cos v, u \sin v, v),$$

have equal Gaussian curvature at the points $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(u, v)$ but that the mapping $\bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.