Math 433 W 2019 - Week 12-13,

HW5 and 6 merged, due on Apr 17, Wednesday

Lesson plan, reading assignments and homework:

Week 12-13: Sections 4-4, 4-5, 4-6, 4-7 in do Carmo's book, geodesics and Gauss-Bonnet Theorem. There are 10 questions totally counting 200 points, since they are the union of two standard homework sets.

You'll also find Hitchin's notes Page 65-75 very useful. Especially Page 74 the geodesic equations. These two weeks is the 1st step in Riemannian geometry. Almost everything we talk in class has generalizations in any higher-dimensional Riemannian manifolds.

1.[20pts] For $\{0 < u < 2\pi, a < v < b\} \subset \mathbb{R}^2$, the map

$$\sigma(u, v) = (f(v)\cos u, f(v)\sin u, g(v)) \subset \mathbb{R}^3$$

is a surface of revolution. Assume the profile curve on the xz plane $C = (f(v), g(v)), \quad a < v < b, \quad f(v) > 0$, is arc-length parametrized, i.e. $f_v^2 + g_v^2 = 1$. We have

$$I = f(v)^2 du^2 + dv^2.$$

a)show that the geodesic equation in this case is

$$\ddot{v} = f(v)\frac{df}{dv}(\dot{u})^2,$$

and

$$\frac{d}{dt}(f^2(v)\dot{u}) = 0.$$

b)show that the meridian $u \equiv C$ is a geodesic.

c)show that a parallel $v \equiv C$ is a geodesic iff $f_v(C) = 0$.

d)In the torus $\sigma(u, v) = ((100 + \cos v)\cos u, (100 + \cos v)\sin u, \sin v) \subset \mathbb{R}^3$, there's a circle $z = 1$ on top. Is it a geodesic?

Solution:

a) By the geodesic equation (in Hitchin's notes):

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

We have $E = f(v)^2, F = 0, G = 1$, and plugging in the above equations we immediately see they becomes

$$\ddot{v} = f(v)\frac{df}{dv}(\dot{u})^2,$$

and

$$\frac{d}{dt}(f^2(v)\dot{u}) = 0.$$

b) For the meridian $u \equiv C$, $\frac{d}{dt}(f^2(v)\dot{u})$ always hold. And we have the curve being arc-length, which means

$$f(v)^2\dot{u}^2 + \dot{v}^2 = 1.$$

It is easy to see that this means $\dot{v} = \pm 1$. Then we check $\ddot{v} = f(v)\frac{df}{dv}(\dot{u})^2$ also holds.
c)A parallel $v = C$ is a geodesic if and only if $f_v = 0$ when $v = C$. If $v = C$, $f(v)^2\dot{u}^2 + \dot{v}^2 = 1$ implies $\dot{v} \neq 0$ is a constant, so $\frac{d}{dt}(f^2(v)\dot{u})$ holds. We still need to check $\ddot{v} = f(v)\frac{df}{dv}(\dot{u})^2$. And it holds if and only if $f_v(C) = 0$. Hence the conclusion.
d)No. $k_g \neq 0$, See question 2.

Remark: Besides doing the calculation of geodesic equations, we can try to use the tangent cone idea to visualize whether or not a parallel is a geodesic.
Here's an online demo of parallel transport on the sphere, and your browser needs to support Java: http://torus.math.uiuc.edu/jms/java/dragsphere/

As we mentioned in the March 24th class, if we parallel transport a vector along a non-geodesic closed loop in S^2(indeed we can do in any surface), the vector does not get back to itself. However, if we parallel transport a vector along a geodesic loop, the vector will come back to itself.

2. [20pts] Compute the geodesic curvature of the upper parallel (circle $z = 1$) of the torus $\sigma(u, v) = ((100 + \cos v)\cos u, (100 + \cos v)\sin u, \sin v) \subset \mathbb{R}^3$.
Solution:
$k_g = 1/100$. Since the normal vector of the surface is perpendicular to the normal of the curve, by Euler's theorem $k_g = k = 1/100$.

Before we do 3, here's a simple definition we now introduce (see page 147 in do Carmo):
If a regular connected curve C on S satisfies this property: for all $p \in C$, the tangent line of C is a principal direction at p; then C is said to be a **line of curvature** of S.

3. a)Show that if a geodesic ($k \neq 0$ everywhere) is also a line of curvature, then it is a plane curve. (Hint: assume arc-length, use the local frame.)
b) Give an example of a line of curvature that is not a geodesic.

a)Suppose the curve is arc-length parameterized by $\gamma(t)$.
Now $\vec{t}, \vec{n}, \vec{b}$ denote the Frenet frame, and let $\vec{N}$ denote the normal vector of the surface. Firstly we know that being a geodesic implies that $k = k_n$ and $\vec{n} = \pm\vec{N}$. Then a line of curvature means that $d\vec{N}/dt = -k_n\vec{t}$, since the tangent lines is a principal direction. Now we know the torsion τ is always

0, because by definition it is $d\vec{n}/dt \cdot \vec{b} = \pm d\vec{N}/dt \cdot \vec{b} = \mp k_n \vec{t} \cdot \vec{b} = 0$. Since $k \neq 0$ everywhere and $\tau \equiv 0$, we know it is a plane curve.

b) No. Because any curve on a plane is a line of curvature. And only straight lines are geodesics on the plane.

4. [10pts] Let $\vec{v}, \vec{w}$ be tangent vector fields along a curve $\gamma : (a, b) \to S$. Show that

$$\frac{d}{dt} < v(\vec{t}), w(\vec{t}) >=< \nabla_\gamma v(t), w(\vec{t}) > + < v(\vec{t}), \nabla_\gamma w(t) > .$$

[Hint: use the definition of the covariant derivative and the computation should be very easy.]

Solution:

$$\frac{d}{dt} < v(\vec{t}), w(\vec{t}) >=< \frac{d}{dt} v(\vec{t}), w(\vec{t}) > + < v(\vec{t}), \frac{d}{dt} w(\vec{t}) >$$

$$=< \nabla_\gamma v(t) + (\frac{d}{dt} v(\vec{t}) \cdot \vec{n})\vec{n}, w(\vec{t}) > + < v(\vec{t}), \nabla_\gamma w(t) + (\frac{d}{dt} w(\vec{t}) \cdot \vec{n})\vec{n} >=< \nabla_\gamma v(t), w(\vec{t}) > + < v(\vec{t}), \nabla_\gamma w(t) > .$$

5. [20pts] Show that geodesic circles on constant Gaussian curvature surfaces have constant geodesic curvature.

Solution: A simple solution is the following:

There exist a coordinate (geodesic polar (r, θ), for example), such that $E = 1, F = 0, G = C$ and $G_r = c$ in geodesic circles. On do Carmo p.256, by Proposition 4 (Theorem of Liouville), curvature of curve $r \equiv A$, $k_g = \frac{G_r}{2G\sqrt{E}}$. Hence k_g is constant.

6. [10pts] In geodesic normal coordinates $I = du^2 + G(u, v)dv^2$. Show that

$$K = -\frac{1}{\sqrt{G}} \frac{\partial^2 \sqrt{G}}{\partial u^2}.$$

[Hint:] Start with $K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}})$, if you get stuck, compute $\frac{\partial \sqrt{G}}{\partial u}$ and you'll see the pattern.

Solution:

$K = -\frac{1}{2\sqrt{EG}}(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}})$.

Now let $E = 1, F = 0$, $K = \frac{1}{2\sqrt{G}}\frac{\partial}{\partial u}\frac{G_u}{\sqrt{G}}$.

We have $\frac{\partial \sqrt{G}}{\partial u} = G_u \frac{1}{2}\frac{1}{\sqrt{G}}$. Now product rule give the answer: $K = -\frac{1}{\sqrt{G}}\frac{\partial^2 \sqrt{G}}{\partial u^2}$.

7. [10pts] Geodesics are solutions to the 1-dimensional variational problem. Minimal surfaces are the 2-dimensional analog of geodesics because it's the local area minimizer. A surface in $\mathbb{R}^3$ is called a minimal surface if it has zero mean curvature at every point.

a)Show that a minimal surface must have non-positive Gaussian curvature everywhere. (Consider principal curvatures.)

b)Show that there's no compact minimal surface embedded in $\mathbb{R}^3$. (Recall a fact that there must be a point on the compact surface in $\mathbb{R}^3$ with positive Gaussian curvature.)

Solution: a) as we mentioned in class, k_1, k_2 are two principal curvatures. $2H = k_1 + k_2 = 0$, this means $k_1 k_2 \leq \frac{1}{4}(k_1 + k_2) = 0$.
b)Suppose there's a compact minimal surface embedded in $\mathbb{R}^3$, then there must be a point on the compact surface in $\mathbb{R}^3$ with positive Gaussian curvature. Contradict against part a).
Here's some illustration of Minimal Surfaces. We actually see them a lot in daily life, for example, the shape of a soap film.
http://torus.math.uiuc.edu/jms/Images/almgren/

8.(do Carmo 4-5, Q1:)[30pts]

> Let $S \subset R^3$ be a regular, compact, connected, orientable surface which is not homeomorphic to a sphere. Prove that there are points on S where the Gaussian curvature is positive, negative, and zero.

[Hint:] Use global Gauss-Bonnet, and you'll again find the following fact useful: there must be a point on a compact surface in $\mathbb{R}^3$ with positive Gaussian curvature.

Solution:

The surface is not a sphere, its Euler number ≤ 0. Since the surface is compact in $\mathbb{R}^3$, there is at least one point with positive Gaussian curvature.
Gaussian curvature is a smooth function on the surface, and hence it's positive in at least a small open set around that point. It has to be somewhere negative because otherwise its integral would be positive, in contradiction to Gauss-Bonnet.
Now take a path connecting a positive point with a negative point. The Gaussian curvature is a continuous function on this path(think as a closed interval). By the intermediate value theorem, there must exist a point in this path with zero Gaussian curvature.

The following question are indeed in the textbook Page 280

9. [30pts] Let $S \subset \mathbb{R}^3$ be a surface with Gaussian curvature $K \leq 0$. Show that two geodesics γ_1 and γ_2 on S which start at a point p cannot meet again at a point q in such a way that together they bound a region S' on S which is homeomorphic to a disk.
[Hint:] Local Gauss-Bonnet will help.

Solution:

If there are two geodesics γ_1 and γ_2 pass through p, q, then they form a 2-gon since S' is is homeomorphic to a disk. Let θ_1, θ_2 be the two angles at p and q. Apply local Gauss-Bonnet to this 2-gon,

$$\iint_{S'} K dA + \pi - \theta_1 + \pi - \theta_2 = 2\pi,$$

which means $\theta_1 + \theta_2 = \iint_{S'} K dA \leq 0$.
Hence $\theta_1 + \theta_2 = 0$, contradicting to the assumption that there are two geodesics (think about the exponential map).

10. [30pts] Let $S \subset \mathbb{R}^3$ be a surface homeomorphic to a cylinder and with Gaussian curvature $K < 0$. Show that S has at most one simple closed geodesic.
[Hint:] Q9 will help.

Solution:

Suppose there are two simple closed geodesics on S, γ_1 and γ_2. Of course, there are 3 possibilities, either γ_1 and γ_2 dont intersect, they intersect in one point, or they intersect in more than one point. If they intersect in more than one point, then, looking at two adjacent intersection points (as viewed traveling along, say, γ_1), the region bounded by γ_1 and γ_2 is homeomorphic to a disc, since γ_1 and

γ_2 are simple closed curves. However, by Q9, this is impossible, since $K < 0$ on S, so we see that γ_1 and γ_2 can intersect in at most 1 point. On the other hand, suppose γ_1 and γ_2 dont intersect at all. Let S' be the region bounded by γ_1 and γ_2 . Then, since γ_1 and γ_2 are simple closed curves and S is homeomorphic to a cylinder, S' is also homeomorphic to a cylinder. We have

$$2\pi\chi(S') = 0 = \iint_{S'} K dA < 0,$$

contradiction.

Thus, we conclude that γ_1 and γ_2 must intersect in exactly one point. However, for this to be the case, γ_1 and γ_2 must be tangent at their point of intersection, which, by the uniqueness of geodesics, implies that γ_1 and γ_2 describe the same curve. Therefore, we conclude that S has at most one simple closed geodesic.

Note 1: The example for this question is the pseudosphere, which is homeomorphic to the cylinder and having negative curvature everywhere.

The rest questions are suggested HW problems, solutions provided.

1.(Counterpart of do Carmo 4-5, Q1:)

If the surface $S \subset \mathbb{R}^3$ is not assumed to be compact, are there still always points with $K = 0, K > 0, K < 0$?

Solution: Clearly not true. Think about the plane, cylinder, pseudosphere etc. When compact condition is removed, many good properties fail to be true. See the above Note 2 to Q 10.

2.(do Carmo 4-5, Q2:)

Let T be a torus of revolution. Describe the image of the Gauss map of T and show, without using the Gauss-Bonnet theorem, that

$$\iint_T K \, d\sigma = 0.$$

Compute the Euler-Poincaré characteristic of T and check the above result with the Gauss-Bonnet theorem.

Solution: The Gauss map part of this question we refer to the Q6 in the sample final. Basically, the Gauss map on the torus will cover the unit sphere twice: once by the non-positive part and once by the non-negative part of Gaussian curvature.

The Gauss-Bonnet theorem part of the question we refer to Exam 2 the last question.

3.(do Carmo 4-5, Q4:)

Compute the Euler-Poincaré characteristic of

a. An ellipsoid.

b. The surface $S = \{(x, y, z) \in R^3; x^2 + y^{10} + z^6 = 1\}$.

Solution: Denote the ellipsoid by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

do the change of variable $p = \frac{x}{a}, q = \frac{y}{b}, r = \frac{z}{c}$. This is a homeomorphism to the sphere and hence Euler number of ellipsoid is 2.

For the second surface, do the change of variable $p = x, q = y^5, r = z^3$. This is a homeomorphism to the sphere and hence Euler number of surface S is 2.

4.(do Carmo 4-6, Q9:)

(*A Local Isoperimetric Inequality for Geodesic Circles.*) Let $p \in S$ and let $S_r(p)$ be a geodesic circle of center p and radius r. Let L be the arc length of $S_r(p)$ and A be the area of the region bounded by $S_r(p)$. Prove that

$$4\pi A - L^2 = \pi^2 r^4 K(p) + R,$$

where $K(p)$ is the Gaussian curvature of S at p and

$$\lim_{r \to 0} \frac{R}{r^4} = 0.$$

Thus, if $K(p) > 0$ (or < 0) and r is small, $4\pi A - L^2 > 0$ (or < 0). (Compare the isoperimetric inequality of Sec. 1-7.)

Note: You can actually find a solution in stackexchange of this question using geodesic polar coordinates. However that solution has serious gaps (as in the solution on stackexchange to some other questions, Q8 for example). We'll not have geodesic polar coordinates in the final. If you are interested in this question, we can talk in office hours.

5.(do Carmo 4-6, Q13:)

(*The Holonomy Group.*) Let S be a regular surface and $p \in S$. For each piecewise regular parametrized curve $\alpha: [0, l] \to S$ with $\alpha(0) = \alpha(l) = p$, let $P_\alpha: T_p(S) \to T_p(S)$ be the map which assigns to each $v \in T_p(S)$ its parallel transport along α back to p. By Prop. 1 of Sec. 4-4, P_α is a linear isometry of $T_p(S)$. If $\beta: [l, \bar{l}]$ is another piecewise regular parametrized curve with $\beta(l) = \beta(\bar{l}) = p$, define the curve $\beta \circ \alpha: [0, \bar{l}] \to S$ by running successively first α and then β; that is, $\beta \circ \alpha(s) = \alpha(s)$ if $s \in [0, l]$, and $\beta \circ \alpha(s) = \beta(s)$ if $s \in [l, \bar{l}]$.

a. Consider the set

$$H_p(S) = \{P_\alpha: T_p(S) \to T_p(S); \text{ all } \alpha \text{ joining } p \text{ to } p\},$$

where α is piecewise regular. Define in this set the operation $P_\beta \circ P_\alpha = P_{\beta\circ\alpha}$; that is, $P_\beta \circ P_\alpha$ is the usual composition of performing first P_α and then P_β. Prove that, with this operation, $H_p(S)$ is a group (actually, a subgroup of the group of linear isometries of $T_p(S)$). $H_p(S)$ is called the *holonomy group* of S at p.

b. Show that the holonomy group at any point of a surface homeomorphic to a disk with $K \equiv 0$ reduces to the identity.

c. Prove that if S is connected, the holonomy groups $H_p(S)$ and $H_q(S)$ at two arbitrary points $p, q \in S$ are isomorphic. Thus, we can talk about *the* (abstract) *holonomy group of a surface*.

d. Prove that the holonomy group of a sphere is isomorphic to the group of 2×2 rotation matrices (cf. Exercise 22, Sec. 4-4).

Note: This is definitely not examinable, and hence in the solution below we'll use standard Riemannian Geometry notion, see the notes for Further reading.

$$R(\partial_1, \partial_2)\partial_1 = \frac{-4}{(1+||u||^2)^2}\partial_2\,, \quad R(\partial_1, \partial_2)\partial_2 = \frac{4}{(1+||u||^2)^2}\partial_1\,.$$

Thus we can write in matrix form:

$$R(\partial_1, \partial_2) = \frac{4}{(1+||u||^2)^2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We see that $R(X, Y) = (X_1Y_2 - X_2Y_1)R(\partial_1, \partial_2)$, where X and Y are two tangent vectors at $p = (u_1, u_2)$. The Lie algebra generated by such matrices is precisely the Lie algebra of $SO(2)$. Since the sphere is simply connected and $SO(2)$ is connected, we know that the holonomy group lies within $SO(2)$. But each element m of $SO(2)$ can be written as

$$m = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

Therefore, the holonomy group is precisely $SO(2)$.

6.(do Carmo 4-4, Q3:)

Verify that the surfaces

$$\mathbf{x}(u, v) = (u\cos v, u\sin v, \log u), \qquad u > 0$$
$$\bar{\mathbf{x}}(u, v) = (u\cos v, u\sin v, v),$$

have equal Gaussian curvature at the points $\mathbf{x}(u, v)$ and $\bar{\mathbf{x}}(u, v)$ but that the mapping $\bar{\mathbf{x}} \circ \mathbf{x}^{-1}$ is not an isometry. This shows that the "converse" of the Gauss theorem is not true.

Solution: We will denote σ and $\bar{\sigma}$for the two parametriztions. $\sigma_u = (\cos v; \sin v; u^{-1})$; $\sigma_v = (-u\sin v; u\cos v; 0)$ and

$$E = 1 + u^{-2}; F = 0, G = u^2$$

Because $F = 0$ we have the formula

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right) = \frac{-1}{(1+u^2)^2}.$$

Also, we have $\bar{\sigma}_u = (\cos v; \sin v; 0)$; $\bar{\sigma}_v = (-u\sin v; u\cos v; 0)$ and and

$$\bar{E} = 1; \bar{F} = 0, \bar{G} = 1 + u^2.$$

Using the same formula we have

$$\bar{K} = -\frac{1}{2\sqrt{\bar{E}\bar{G}}}\left(\frac{\partial}{\partial u}\frac{\bar{G}_u}{\sqrt{\bar{E}\bar{G}}} + \frac{\partial}{\partial v}\frac{\bar{E}_v}{\sqrt{\bar{E}\bar{G}}}\right) = \frac{-1}{(1+u^2)^2}.$$

Therefore the surfaces have the same curvature. If the map $\sigma \circ \bar{\sigma}^{-1}$ were an isometry then by Proposition 1, page 220 in do Carmo, we would have

$$E = \bar{E}, F = \bar{F}, G = \bar{G}$$

. Because this is false the surfaces are not isometric.