Math 433 W 2019 - HW2

Due: Wed, Feb 6, 11:00 am

Lesson plan, reading assignments and homework:

Week 3, 4: Sections 1-5, 1-6, 1-7 in do Carmo's book.

This HW contains 5+1 problems (one bonus problem), mostly from do Carmo's book(some question numbers are not listed because they are numbered differently in different editions).

100 total=5 $\times$ 20 points. 20 extra.

1.. (Helix is the fundamental example of space curves)

Given the parametrized curve (helix)

$$\alpha(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \quad s \in R,$$

where $c^2 = a^2 + b^2$,

a. Show that the parameter s is the arc length.

b. Determine the curvature and the torsion of α.

c. Determine the osculating plane of α.

d. Show that the lines containing $n(s)$ and passing through $\alpha(s)$ meet the z axis under a constant angle equal to $\pi/2$.

e. Show that the tangent lines to α make a constant angle with the z axis.

2.. (Compute torsion for an arc-length curve)

Show that the torsion τ of α is given by

$$\tau(s) = -\frac{\alpha'(s) \wedge \alpha''(s) \cdot \alpha'''(s)}{|k(s)|^2}.$$

3.. (Warning: this parametrization is not by arc-length)

Compute the curvature of the ellipse

$$x = a\cos t, \quad y = b\sin t, \quad t \in [0, 2\pi], a \neq b,$$

and show that it has exactly four vertices, namely, the points $(a, 0)$, $(-a, 0)$, $(0, b)$, $(0, -b)$.

4. (this gives an alternative interpretation of curvature)

Let C be a plane curve and let T be the tangent line at a point $p \in C$. Draw a line L parallel to the normal line at p and at a distance d of p (Fig. 1-36). Let h be the length of the segment determined on L by C and T (thus, h is the "height" of C relative to T). Prove that

$$|k(p)| = \lim_{d \to 0} \frac{2h}{d^2},$$

where $k(p)$ is the curvature of C at p.

5. (this is a first example of a curvature comparison principle)

If a closed plane curve C is contained inside a disk of radius r, prove that there exists a point $p \in C$ such that the curvature k of C at p satisfies $|k| \geq 1/r$.

6, bonus extra 20 points. (Review of Green's Theorem and preparation for change of variable of surfaces.)

Green's theorem in the plane is a basic fact of calculus and can be stated as follows. Let a simple closed plane curve be given by $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$. Assume that α is positively oriented, let C be its trace, and let R be the interior of C. Let $p = p(x, y)$, $q = q(x, y)$ be real functions with continuous partial derivatives p_x, p_y, q_x, q_y. Then

$$\iint_R (q_x - p_y)\, dx\, dy = \int_C \left(p\frac{dx}{dt} + q\frac{dy}{dt} \right) dt, \tag{9}$$

where in the second integral it is understood that the functions p and q are restricted to α and the integral is taken between the limits $t = a$ and $t = b$. In parts a and b below we propose to derive, from Green's theorem, a formula for the area of R and the formula for the change of variables in double integrals (cf. Eqs. (1) and (7) in the text).

a. Set $q = x$ and $p = -y$ in Eq. (9) and conclude that

$$A(R) = \iint_R dx\, dy = \frac{1}{2}\int_a^b \left(x(t)\frac{dy}{dt} - y(t)\frac{dx}{dt} \right) dt.$$

b. Let $f(x, y)$ be a real function with continuous partial derivatives and $T\colon R^2 \to R^2$ be a transformation of coordinates given by the functions $x = x(u, v)$, $y = y(u, v)$, which also admit continuous partial derivatives. Choose in Eq. (9) $p = 0$ and q so that $q_x = f$. Apply successively Green's theorem, the map T, and Green's theorem again to obtain

$$\begin{aligned}
&\iint_R f(x, y)\, dx\, dy \\
&\quad = \int_C q\, dy = \int_{T^{-1}(C)} (q \circ T)(y_u u'(t) + y_v v'(t))\, dt \\
&\quad = \iint_{T^{-1}(R)} \left\{ \frac{\partial}{\partial u}((q \circ T)y_v) - \frac{\partial}{\partial v}((q \circ T)y_u) \right\} du\, dv.
\end{aligned}$$

Show that

$$\begin{aligned}
&\frac{\partial}{\partial u}(q(x(u, v), y(u, v))y_v) - \frac{\partial}{\partial v}(q(x(u, v), y(u, v))y_u) \\
&\quad = f(x(u, v), y(u, v))(x_u y_v - x_v y_u) = f\frac{\partial(x, y)}{\partial(u, v)}.
\end{aligned}$$

Put that together with the above and obtain the transformation formula for double integrals:

$$\iint_R f(x, y)\, dx\, dy = \iint_{T^{-1}(R)} f(x(u, v), y(u, v))\frac{\partial(x, y)}{\partial(u, v)}\, du\, dv.$$