Math 433 W 2019 - HW3 due on Feb 27 11am

If you plan to start early, you can do it in preparation of Exam 1, indeed exam question will be easier. Note that Exam 1 is on Feb 20, 11:00(sharp)-11:50, which covers 1.1 through 2.4 in do Carmo's book.

Lesson plan, reading assignments and homework:

Week 5,6,7: Sections 2-2, 2-3, 2-4 in do Carmo's book.

This HW contains 6 problems from do Carmo's book, and you can take a look to prepare for the Exam 1.

Grading: 100points=$40 + 6 \times 10$ points each.

Also, a collection of definition and facts about differential of multi-variable vector valued maps is given in the end. It can be regarded as the notes for the very first lecture of this course, which turns out to be very useful here.

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1. (2-2-1)

Show that the cylinder $\{(x, y, z) \in R^3; x^2 + y^2 = 1\}$ is a regular surface, and find parametrizations whose coordinate neighborhoods cover it.

2. (2-2-7,a,b)

Let $f(x, y, z) = (x + y + z - 1)^2$.

a. Locate the critical points and critical values of f.

b. For what values of c is the set $f(x, y, z) = c$ a regular surface?

3. (2-2-8)

Let $\mathbf{x}(u, v)$ be as in Def. 1. Verify that $d\mathbf{x}_q : R^2 \to R^3$ is one-to-one if and only if

$$\frac{\partial \mathbf{x}}{\partial u} \wedge \frac{\partial \mathbf{x}}{\partial v} \neq 0.$$

4. (2-2-11)

Show that the set $S = \{(x, y, z) \in R^3; z = x^2 - y^2\}$ is a regular surface and check that parts a and b are parametrizations for S:

a. $\mathbf{x}(u, v) = (u + v, u - v, 4uv)$, $(u, v) \in R^2$.

b. $\mathbf{x}(u, v) = (u \cosh v, u \sinh v, u^2)$, $(u, v) \in R^2$, $u \neq 0$.

Which parts of S do these parametrizations cover?

5.(2-4-1)

Determine the tangent planes of $x^2 + y^2 - z^2 = 1$ at the points $(x, y, 0)$ and show that they are all parallel to the z axis.

6.(2-4-3)

Show that the equation of the tangent plane of a surface which is the graph of a differentiable function $z = f(x, y)$, at the point $p_0 = (x_0, y_0)$, is given by

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Recall the definition of the differential df of a function $f: R^2 \to R$ and show that the tangent plane is the graph of the differential df_p.

Collection of useful facts about smooth maps and their differentials.

Definition 1.1.1. *Let U be an open set in $\mathbb{R}^n$, and $f : U \to \mathbb{R}$ a continuous function. The function f is smooth (or C^∞) if it has derivatives of any order.*

Note that not all smooth functions are analytic. For example, the function

$$f(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$

is a smooth function defined on $\mathbb{R}$ but is not analytic at $x = 0$. (Check this!)

Now let U be an open set in $\mathbb{R}^n$ and V be an open set in $\mathbb{R}^m$. Let $f = (f^1, \cdots, f^m) : U \to V$ be a continuous map. We say f is smooth if each component f^i, $1 \le i \le m$, is a smooth function.

Definition 1.1.2. *The* differential *of f, df, assigns to each point $x \in U$ a linear map $df_x : \mathbb{R}^n \to \mathbb{R}^m$ whose matrix is the* Jacobian matrix *of f at x,*

$$df_x = \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \cdots & \frac{\partial f^1}{\partial x^n}(x) \\ \vdots & \vdots & \vdots \\ \frac{\partial f^m}{\partial x^1}(x) & \cdots & \frac{\partial f^m}{\partial x^n}(x) \end{pmatrix}.$$

Now, we are ready to introduce the notion of *diffeomorphism*.

Definition 1.1.3. *A smooth map $f : U \to V$ is a* diffeomorphism *if f is one-to-one and onto, and $f^{-1} : V \to U$ is also smooth.*

Obviously

- If $f : U \to V$ is a diffeomorphism, so is f^{-1}.
- If $f : U \to V$ and $g : V \to W$ are diffeomorphisms, so is $g \circ f$.

As a consequence, we get

Theorem 1.1.4. *If $f : U \to V$ is a diffeomorphism, then at each point $x \in U$, the linear map df_x is an isomorphism. In particular,* $\dim U = \dim V$.

Proof. Applying the chain rule to $f^{-1} \circ f = id_U$, and notice that the differential of the identity map $id_U : U \to U$ is the identity transformation $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$, we get

$$df^{-1}_{f(x)} \circ df_x = \mathrm{Id}_{\mathbb{R}^n}.$$

The same argument applies to $f \circ f^{-1}$, which yields

$$df_x \circ df^{-1}_{f(x)} = \mathrm{Id}_{\mathbb{R}^m}.$$

By basic linear algebra, we conclude that $m = n$ and that df_x is an isomorphism. □

The inverse of the previous theorem is not true. For example, we consider the map

$$f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}, \quad (x^1, x^2) \mapsto ((x^1)^2 - (x^2)^2, 2x^1x^2).$$

Then at each point $x \in \mathbb{R}^2 \setminus \{0\}$, df_x is an isomorphism. However, f is not invertible since $f(x) = f(-x)$. (What is the map f if we identify $\mathbb{R}^2$ with $\mathbb{C}$?)

The inverse function theorem is a partial inverse of the previous theorem, which claims that an isomorphism in the linear category implies a *local* diffeomorphism in the differentiable category.

Theorem 1.1.5 (Inverse Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be an open set, $p \in U$ and $f : U \to \mathbb{R}^n$. If the Jacobian df_p is invertible at p, then there exists a neighbourhood U_p of p and a neighbourhood $V_{f(p)}$ of $f(p)$ such that*

$$f|U_p : U_p \to V_{f(p)}$$

is a diffeomorphism.