Math 433 W 2019 - HW4 solution

Lesson plan, reading assignments and homework:

Week 3: Sections 2-5, 2-6, 3-2, 3-3, 4-2, 4-3 in do Carmo's book.

These materials are a bit long in do Carmo's book. An alternative option is to read (skip everything about complex numbers) Hitchin's notes page 45 to page 80, which is much shorter. We will firstly follow Hitchin's treatment and then add geometric interpretation, (the Gauss map for example).

This HW contains 5 problems plus one bonus problem, and will play a role in our future lecture. 100 total=5×20 points. And 20 points bonus, which is hard.

1.. Consider the stereographic projection of the sphere to the plane, which is inverse of the map $\mathbb{R}^2 \to S^2\setminus$ (North Pole):

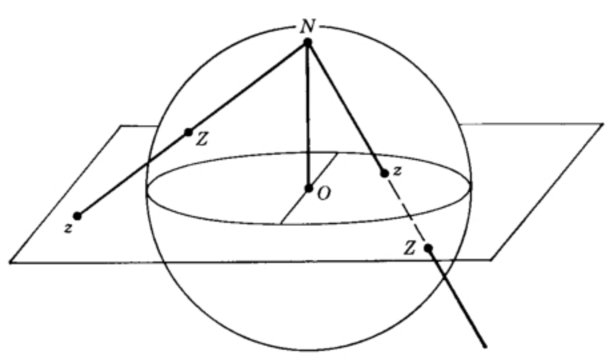
$$(x, y) \mapsto \frac{1}{1+x^2+y^2}(2x, 2y, -1+x^2+y^2),$$

and show that the fundamental form for the sphere in these local coordinates is

$$I_F = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}.$$

Solution. This is a straight forward computation: $\sigma_x = (\frac{2+2y^2-2x^2}{(1+x^2+y^2)^2}, \frac{-4xy}{(1+x^2+y^2)^2}, \frac{4x}{(1+x^2+y^2)^2})$, $\sigma_y = (\frac{-4xy}{(1+x^2+y^2)^2}, \frac{2+2x^2-2y^2}{(1+x^2+y^2)^2}, \frac{4y}{(1+x^2+y^2)^2})$.

Taking $E = \sigma_x \cdot \sigma_x, F = \sigma_x \cdot \sigma_y, G = \sigma_y \cdot \sigma_y$ yields the result. The computation is straight forward: for example, $E = \sigma_x \cdot \sigma_x = \frac{1}{(1+x^2+y^2)^4}[(4+4y^4+4x^4 = 8y^2-8x^2-8x^2y^2)+16y^2+16x^2] = \frac{4(1+x^2+y^2)^2}{(1+x^2+y^2)^4}$. Note that this is a conformal map between $S^2 \setminus (North \quad Pole)$ and the plane.



2. (Characterization of conformal maps) Let $\phi : S \to \bar{S}$ be a diffeomorphism between two surfaces in $\mathbb{R}^3$. Such a map is called conformal if for all $p \in S$, and $v_1, v_2 \in T_p(S)$ (the tangent plane) we have

$$\langle d\phi_p(v_1), d\phi_p(v_2)\rangle = \lambda^2 \langle v_1, v_2\rangle_p$$

for some nowhere-zero function λ.(**note:** the book take this as the definition of a conformal map, while we define angle preserving map as conformal and say the above is a characterization of conformal maps.)
ϕ is said to be angle-preserving, if

$$\cos(v_1, v_2) = \cos(d\phi_p(v_1), d\phi_p(v_2)),$$

which means

$$\frac{\langle v_1, v_2\rangle}{\|v_1\|\|v_2\|} = \frac{\langle d\phi(v_1), d\phi(v_2)\rangle}{\|d\phi(v_1)\|\|d\phi(v_2)\|}$$

Prove that ϕ is locally conformal if and only if it preserves angles.

Solution.
The only if direction, just consider

$$\cos\bar{\theta} = \frac{\langle d\varphi(\alpha'), d\varphi(\beta')\rangle}{|d\varphi(\alpha')||d\varphi(\beta')|} = \frac{\lambda^2\langle \alpha', \beta'\rangle}{\lambda^2|\alpha'||\beta'|} = \cos\theta,$$

And the if direction,
Take e_1, e_2 a set of orthonormal basis of T_pS. Let:

$$< d\phi_p(e_1), d\phi_p(e_1) >= \lambda_1$$
$$< d\phi_p(e_1), d\phi_p(e_2) >= \mu$$
$$< d\phi_p(e_2), d\phi_p(e_2) >= \lambda_2$$

Now take:

$$v_1 = e_1$$
$$v_2 = \cos\theta\ e_1 + \sin\theta\ e_2$$

The equation in the question implies that:

$$\cos\theta = \frac{\lambda_1\cos\theta + \mu\sin\theta}{\sqrt{\lambda_1\left(\lambda_1\cos^2\theta + 2\mu\sin\theta\cos\theta + \lambda_2\sin^2\theta\right)}}$$

Take $\theta = \frac{\pi}{2}$ to get $\mu = 0$. This implies that:

$$\lambda_1 = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta$$

Or $\lambda_1 = \lambda_2$. Hence:

$$\begin{aligned}
\langle d\phi_p(e_1), d\phi_p(e_1)\rangle &= \lambda_1 \langle e_1, e_1\rangle_p \\
\langle d\phi_p(e_2), d\phi_p(e_2)\rangle &= \lambda_1 \langle e_2, e_2\rangle_p \\
\langle d\phi_p(e_1), d\phi_p(e_2)\rangle &= \lambda_1 \langle e_1, e_2\rangle_p \qquad (= 0)
\end{aligned}$$

Since both $\langle , \rangle_p$ and $\langle d\phi_p(), d\phi_p()\rangle$ are bilinear forms, the above is true for all $v_1, v_2 \in T_pS$.

3. (Surface of revolution) We have a lot of examples(sphere, cylinder, hyperboloid, torus, etc) are obtained by rotating a regular connected plane curve C about an axis in the plane which does not intersect the curve. Usually, we take the xz plane as the plane of the curve C and the z-axis as the rotation axis. We parameterize the curve C by

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

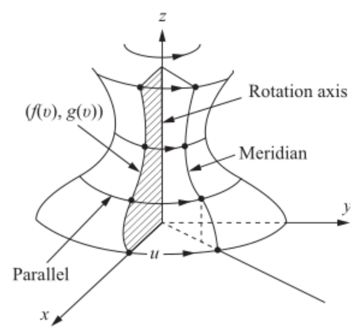
and denote u by the rotation angle about the z-aixs. Then the map

$$\sigma(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$$

is a parameterizations from the open set $\{0 < u < 2\pi, a < v < b\}$ in $\mathbb{R}^2$ to the surface S. Verify that the first and second fundamental forms are

$$I = f(v)^2 du^2 + dv^2, \quad \text{and} \quad (-1)II = f g_v du^2 + (f_v g_{vv} - f_{vv} g_v) dv^2.$$

A picture(from do Carmo's book) is given below:



Solution.

$$\sigma_v = (f_v \cos u, f_v \sin u, g_v); \quad \sigma_u = (-f \sin u; f \cos u; 0):$$

So

$$E = f^2; F = 0; G = f_v^2 + g_v^2 = 1.$$

Hence

$$I = f(v)^2 du^2 + dv^2.$$

Further, we have
$\sigma_u \wedge \sigma_v = (-f g_v \cos u, -f g_v \sin u, f f_v)$; $\quad ||\sigma_u \wedge \sigma_v|| = f$. Hence

$$\vec{n} = (-g_v \cos u, -g_v \sin u, f_v);$$

$$\sigma_{vv} = (f_{vv} \cos u, f_{vv} \sin u, g_{vv});$$

$$\sigma_{uv} = (-f_v \sin u, f_v \cos u, 0);$$

$$\sigma_{uu} = (-f \cos u, -f \sin u, 0).$$

So the second fundamental form $II = -(f g_v du^2 + (f_v g_{vv} - f_{vv} g_v) dv^2)$, where $L = \sigma_{uu} \cdot \vec{n} = f g_v(\cos^2 u + \sin^2 u)$, $M = \sigma_{uv} \cdot \vec{n} = 0$, and $N = \sigma_{vv} \cdot \vec{n} = f_v g_{vv} - f_{vv} g_v$.

4.(Question 3-3-6 in do Carmo.)

(*A Surface with $K \equiv -1$; the Pseudosphere.*)

a. Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).

b. Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

c. Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

Solution:

For part a), check the plane curve given by

$$\alpha(t) : (0, \pi) \to \mathbb{R}^2, t \to (\sin t, \cos t + \log \tan \frac{t}{2}),$$

where t is the angle between y-axis and the tangent vector $\alpha'(t)$.

Part b) is straight forward checking of definition.

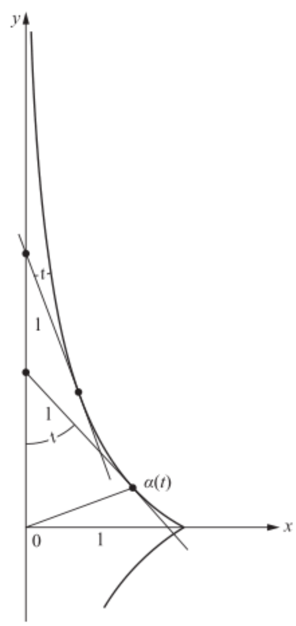


Figure 1-9. The tractrix.

(A)

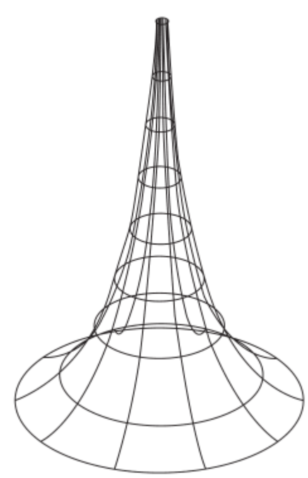


Figure 3-22. The pseudosphere.

(B)

FIGURE 1. Pictures(from do Carmo's book)

Part c): We indeed only deal with the upper half of the tractrix or pseudosphere, since it's symmetric.
Note the curve in part a) is not arc-length parametrized. And there're several different (indeed equivalent) arc-length parametrizations of the tractrix. But if we start with the parametrization in part a), the only way to do arc-length repara is to do $s = -\log \sin t$ or $\sin t = e^{-s}$, since $|\alpha'(t)| = -\log \sin t$.
Then we have $\gamma(s) = (e^{-s}, \sqrt{1-e^{-2s}} - arccosh(e^s)), s \geq 0$.
And in this case (by Q3 and arc-length) the second fundamental form is

$$II = -fg_v du^2 + 0 + (f_v g_{vv} - f_{vv} g_v) dv^2.$$

Then we have

$$K = \frac{LN - M^2}{EG - F^2} = -\frac{g_v(f_v g_{vv} - f_{vv} g_v)}{f},$$

plugging in $f(v) = e^{-v}, g(v) = \sqrt{1-e^{-2v}} - arccosh(e^v)$ yields the result.

5. (How to make a world map.)

Using the parametrization

$$F(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$

of the unit sphere $S^2 \subset \mathbb{R}^3$, find the first fundamental form.

To draw maps of the Earth, one often uses Mercator's projection of the unit sphere minus the date line:

$$(X, Y) = \left(\theta\,,\, \log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right) \in \mathbb{R}^2,$$

where (θ, ϕ) are the longitude and latitude coordinates on the Earth. What does the first fundamental form of the sphere become in the coordinates (X, Y) of the plane? Deduce that Mercator's projection is conformal but not area-preserving.

Solution: For the sphere, we can regard it as a surface of revolution, and use question 3. And we have $I_{S^2} = 1 \cdot d\phi + \cos^2\phi d\theta^2$.

For the plane(or cylinder), we can directly compute its first fundamental form. We have

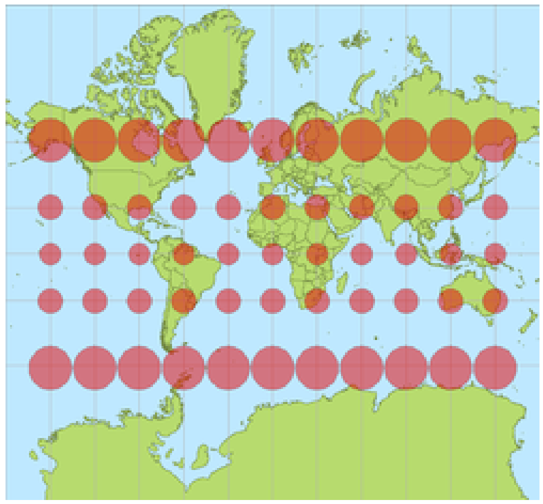
$$\sigma_\theta = (1, 0, 0); \quad \sigma_\phi = (0, \frac{1}{2\tan(\phi/2 + \pi/4)\cos^2(\phi/2 + \pi/4)}, 0);$$

and hence $\sigma_\theta \wedge \sigma_\phi = (0, 0, \frac{1}{2\tan(\phi/2+\pi/4)\cos^2(\phi/2+\pi/4)})$.
Then we just check that

$$I_{\mathbb{R}^2} = \frac{1}{2\tan(\phi/2 + \pi/4)\cos^2(\phi/2 + \pi/4))^2} d\phi^2 + 1 \cdot d\theta^2.$$

Now we have at any point (θ, ϕ), $I_{S^2} = \cos^2\phi(I_{\mathbb{R}^2})$, this is because the $2\tan(\phi/2 + \pi/4)\cos^2(\phi/2 + \pi/4))$ simplified to be $2\sin(\phi/2 + \pi/4)\cos(\phi/2 + \pi/4)) = \sin 2[(\phi/2 + \pi/4)] = \cos\phi$.

Since **the date line ($\theta = 0, \phi \in [-\pi/2, \pi/2]$)is removed, we have** $\cos\phi$ **is a nowhere zero function** on the plane. Hence by Q2 we know this is a conformal map.
To see this is not area-preserving is easy, either one can check the area element $\sqrt{EG - F^2}$ is different for I_{S^2} and $I_{\mathbb{R}^2}$, where on S^2 it is $\cos^2\phi$ and on $\mathbb{R}^2$ it is 1. Or just look at the following picture and note that the red dots have the same area on the sphere but different areas on the plane.



This sort of explains why US and Canada have about the same area, but Canada looks way larger than US in the map.

`Further reading: https://en.wikipedia.org/wiki/Mercator_projection`

6. Bonus, hard, need to know complex number well (The upper half plane)
Consider the unit disk $D = \{x + iy \in \mathbb{C} | x^2 + y^2 < 1\}$ with first fundamental form

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, (compare \quad this \quad to \quad Q1)$$

and the upper half plane $D = \{u + iv \in \mathbb{C} | v > 0\}$ with the first fundamental form

$$\frac{(du^2 + dv^2)}{v^2}.$$

a) Show that there is an isometry form H to D given by

$$w \to z = \frac{w - i}{w + i},$$

where $w = u + iv \in H$ and $z = x + iy \in D$.
b) Show that H with $I = \frac{(du^2+dv^2)}{v^2}$ has constant Gaussian curvature -1.
c)(hard) Compare with Q1, D could be thought as a subset of the unit sphere, which has constant Gaussian curvature $+1$. But we just found an isometry form H to D, which means D has Gaussian curvature -1. Is there any contradiction?
d)(harder) The upper half plane and the pseudosphere as in Q4 both has constant $K = -1$. Can you build some relation between them?

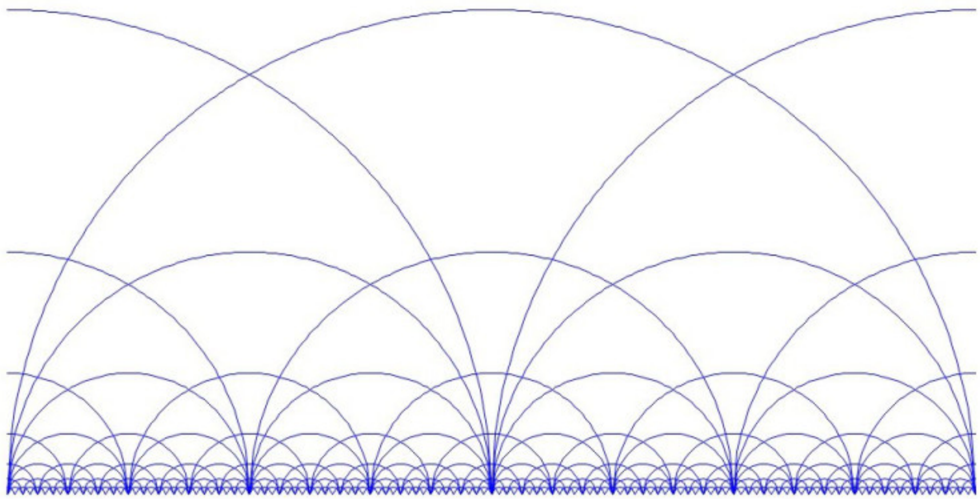


FIGURE 2. Geodesics in H, which we will mention in the future.

Solution:

a) This part could be found at Hitchin's note page 59:
Firstly, denote $|dz|^2 = dx^2 + dy^2$ and $|dw|^2 = du^2 + dv^2$, if $w = f(z)$ where $f(z)$ is a holormophic function, then we have the following claim: $|dw|^2 = |f'(z)|^2|dz|^2$ or we write

$$|dw|^2 = {\frac{|dw|^2}{|dz|^2}}^2 |dz|^2.$$

Then we check that $w \to z = \frac{w-i}{w+i}$, and its inverse $z \to w = \frac{z+i}{z-i}$ are both holomorphic functions.
Then since $I_H = |dw|^2/v^2$, we can change variable $|dw|^2$ by ${\frac{|dw|^2}{|dz|^2}}^2|dz|^2$, where $f'(z) = \frac{2i}{(w+i)}$.
And we also compute that $f'(z)^2 = \frac{-4}{v^2(w+i)^2} = \frac{4\cdot(2i)^2}{(w-\bar{w})((w+i)^2)} = \frac{4}{(1-z)(1-\bar{z})}$.
Hence under transformation $w \to z = \frac{w-i}{w+i}$, $|dw|^2$ becomes $\frac{4}{(1-z)(1-\bar{z})}|dz|^2 = \frac{4(dx^2+dy^2)}{(1-x^2-y^2)^2} = I_D$. This means the map $z = \frac{w-i}{w+i}$ is an isometry.
To verify $|dw|^2 = {\frac{|dw|^2}{|dz|^2}}^2|dz|^2$, one just do a direct computation from the Cauchy-Riemann equation, which is $f'(z) = u_x + iv_x = u_y - iv_y$.

b) There's no $dudv$ in I_H, which means H has an orthogonal parametrization. The Gaussian curvature is given by

$$K = -\frac{1}{2\sqrt{EG}}\left(\frac{\partial}{\partial u}\frac{G_u}{\sqrt{EG}} + \frac{\partial}{\partial v}\frac{E_v}{\sqrt{EG}}\right)$$

Here $E = G = \frac{1}{v^2}$ as given in I_H, and direct computation yeilds $K = -1$.

c) There's no contradiction. The statement "D could be thought as a subset of the unit sphere, which has constant Gaussian curvature +1" is indeed false. There's no isometry from D with the above I_D to any subset of a sphere with the 1st fundamental form in $Q1$.

Some further remark: Locally we are able to find some smooth invertible map making an open region of the unit sphere having Gaussian curvature -1, but that cannot be done for the whole sphere. We'll further explain this in Gauss-Bonnet theorem.

d) The half pseudosphere(as the picture shown in Q4) of curvature -1 is covered by the portion of the hyperbolic upper half-plane with $y \geq 1$.
The covering map is periodic in the x direction of period 2π, and takes the line (in H it indeed called "horocycles") $y = c$ to the meridians of the pseudosphere and the vertical geodesics $x = c$ to the tractrices that generate the pseudosphere.
More explicitly, the map is

$$(x, y) \mapsto (arcosh(y)v \cos x, arcosh(y)v \sin x, arcosh(y)u)$$

and the tractrix is reparemetrized as

$$t \mapsto (u(t) = t - \tanh t, v(t) = \frac{1}{\cosh t}).$$