Math 433 W 2019 - HW4

Due: Wed, Mar 20, 11:00 am

Lesson plan, reading assignments and homework:

Week 3: Sections 2-5, 2-6, 3-2, 3-3, 4-2, 4-3 in do Carmo's book.

These materials are a bit long in do Carmo's book. An alternative option is to read (skip everything about complex numbers) Hitchin's notes page 45 to page 80, which is much shorter. We will firstly follow Hitchin's treatment and then add geometric interpretation, (the Gauss map for example).

This HW contains 5 problems plus one bonus problem, and will play a role in our future lecture.

100 total=5×20 points. And 20 points bonus, which is hard.

1.. Consider the stereographic projection of the sphere to the plane, which is inverse of the map $\mathbb{R}^2 \to S^2 \setminus$ (North Pole):

$$(x, y) \mapsto \frac{1}{1+x^2+y^2}(2x, 2y, -1+x^2+y^2),$$

and show that the fundamental form for the sphere in these local coordinates is

$$I_F = \frac{4(dx^2+dy^2)}{(1+x^2+y^2)^2}.$$

2. (Characterization of conformal maps) Let $\phi : S \to \bar{S}$ be a diffeomorphism between two surfaces in $\mathbb{R}^3$. Such a map is called conformal if for all $p \in S$, and $v_1, v_2 \in T_p(S)$ (the tangent plane) we have

$$\langle d\phi_p(v_1), d\phi_p(v_2)\rangle = \lambda^2 \langle v_1, v_2\rangle_p$$

for some nowhere-zero function λ.(**note:** the book take this as the denomination of a conformal map, while we define angel preserving map as conformal and say the above is a characterization of conformal maps.)

ϕ is said to be angle-preserving, if

$$\cos(v_1, v_2) = \cos(d\phi_p(v_1), d\phi_p(v_2)),$$

which means

$$\frac{\langle v_1, v_2\rangle}{\|v_1\|\|v_2\|} = \frac{\langle d\phi(v_1), d\phi(v_2)\rangle}{\|d\phi(v_1)\|\|d\phi(v_2)\|}$$

Prove that ϕ is locally conformal if and only if it preserves angles.

Hint: the only if direction, just consider

$$\cos\bar{\theta} = \frac{\langle d\varphi(\alpha'), d\varphi(\beta')\rangle}{|d\varphi(\alpha')||d\varphi(\beta')|} = \frac{\lambda^2\langle \alpha', \beta'\rangle}{\lambda^2|\alpha'||\beta'|} = \cos\theta,$$

and the if direction,

Take e_1, e_2 a set of orthonormal basis of T_pS. Let:

$$< d\phi_p(e_1), d\phi_p(e_1) >= \lambda_1$$

$$< d\phi_p(e_1), d\phi_p(e_2) >= \mu$$

$$< d\phi_p(e_2), d\phi_p(e_2) >= \lambda_2$$

Now take:

$$v_1 = e_1; \quad v_2 = \cos\theta\ e_1 + \sin\theta\ e_2$$

The equation in your question implies that:

$$\cos\theta = \frac{\lambda_1\cos\theta + \mu\sin\theta}{\sqrt{\lambda_1\left(\lambda_1\cos^2\theta + 2\mu\sin\theta\cos\theta + \lambda_2\sin^2\theta\right)}}$$

3. (Surface of revolution) We have a lot of examples(sphere, cylinder, hyperbolid, torus, etc) are obtained by rotating a regular connected plane curve C about an axis in the plane which does not intersect the curve. Usually, we take the xz plane as the plane of the curve C and the z-axis as the rotation axis. We parametrize the curve C by

$$x = f(v), \quad z = g(v), \quad a < v < b, \quad f(v) > 0,$$

and denote u by the rotation angle about the z-aixs. Then the map

$$\sigma(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$$

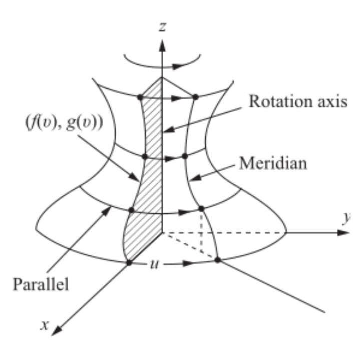
is a parameterization from the open set $\{0 < u < 2\pi, a < v < b\}$ in $\mathbb{R}^2$ to the surface S.

We need to assume the curve C is arc-length parametrized, i.e. $f_v^2 + g_v^2 = 1$.

Verify that the first and second fundamental forms are

$$I = f(v)^2du^2 + dv^2, \quad \text{and} \quad II = fg_vdu^2 + (f_vg_{vv} - f_{vv}g_v)dv^2.$$

A picture(from do Carmo's book) is given below:



4.(Question 3-3-6 in do Carmo.)

(*A Surface with* $K \equiv -1$; *the Pseudosphere.*)

a. Determine an equation for the plane curve C, which is such that the segment of the tangent line between the point of tangency and some line r in the plane, which does not meet the curve, is constantly equal to 1 (this curve is called the *tractrix*; see Fig. 1-9).

b. Rotate the tractrix C about the line r; determine if the "surface" of revolution thus obtained (the *pseudosphere*; see Fig. 3-22) is regular and find out a parametrization in a neighborhood of a regular point.

c. Show that the Gaussian curvature of any regular point of the pseudosphere is -1.

Hint: For part c), we can try to use **need to be careful about the parametrization of the tractrix** the results for surface of revaluation as in question 3. For part a), check the plane curve given by

$$\alpha(t) : (0, \pi) \to \mathbb{R}^2, t \to (\sin t, \cos t + \log \tan \frac{t}{2}),$$

where t is the angle between y-axis and the tangent vector $\alpha'(t)$.

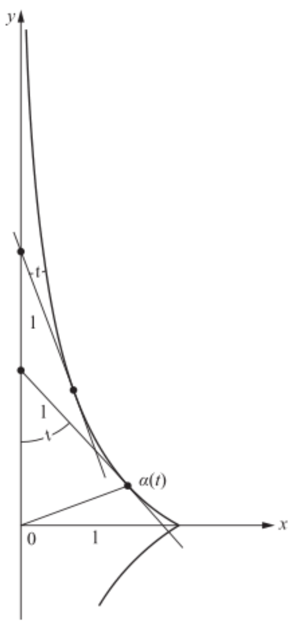


Figure 1-9. The tractrix.

(A)

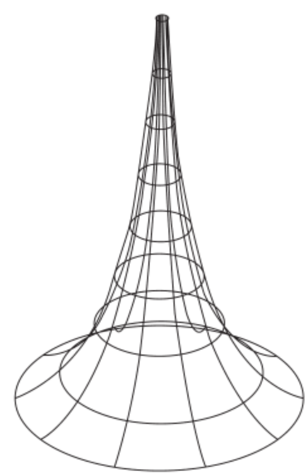


Figure 3-22. The pseudosphere.

(B)

FIGURE 1. Pictures(from do Carmo's book)

5. (How to make a world map.)

Using the parametrization

$$F(\theta, \phi) = (\cos\theta\cos\phi, \sin\theta\cos\phi, \sin\phi)$$

of the unit sphere $S^2 \subset \mathbb{R}^3$, find the first fundamental form.

To draw maps of the Earth, one often uses Mercator's projection of the unit sphere minus the date line:

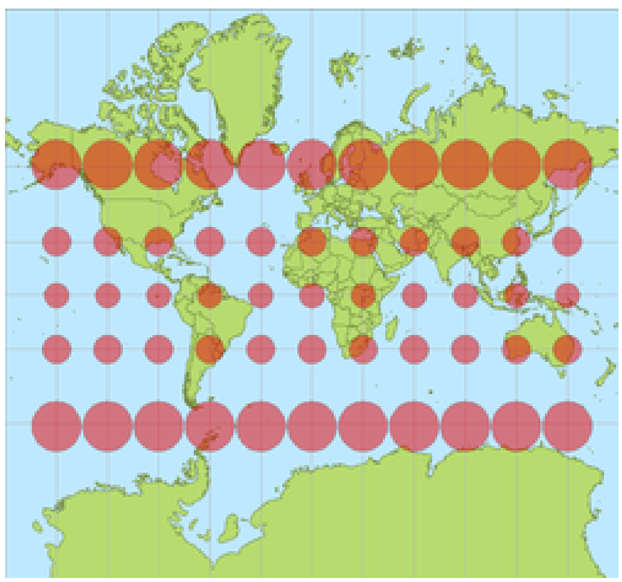
$$(X, Y) = \left(\theta\,,\, \log\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right) \in \mathbb{R}^2,$$

where (θ, ϕ) are the longitude and latitude coordinates on the Earth. What does the first fundamental form of the sphere become in the coordinates (X, Y) of the plane? Deduce that Mercator's projection is conformal but not area-preserving.

Don't convert things to x-y coordinates(that will cause a lot of trouble), we can directly do it in $\theta - \phi$ coordinates.

Hint: For the sphere, we can regard it as a surface of revolution, and use question 3.

For the plane(or cylinder), we can directly compute its first fundamental form.



Further reading: `https://en.wikipedia.org/wiki/Mercator_projection`

6. Bonus, hard, need to know complex number well (The upper half plane)

Consider the unit disk $D = \{x + iy \in \mathbb{C} | x^2 + y^2 < 1\}$ with first fundamental form

$$\frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2}, (compare \quad this \quad to \quad Q1)$$

and the upper half plane $D = \{u + iv \in \mathbb{C} | v > 0\}$ with the first fundamental form

$$\frac{(du^2 + dv^2)}{v^2}.$$

a) Show that there is an isometry form H to D given by

$$w \to z = \frac{w-i}{w+i},$$

where $w = u + iv \in H$ and $z = x + iy \in D$.

b) Show that H with $I = \frac{(du^2+dv^2)}{v^2}$ has constant Gaussian curvature -1.

c)(hard) Compare with Q1, D could be thought as a subset of the unit sphere, which has constant Gaussian curvature $+1$. But we just found an isometry form H to D, which means D has Gaussian curvature -1. Is there any contradiction?

d)(harder) The upper half plane and the pseudosphere as in Q4 both has constant $K = -1$. Can you build some relation between them?

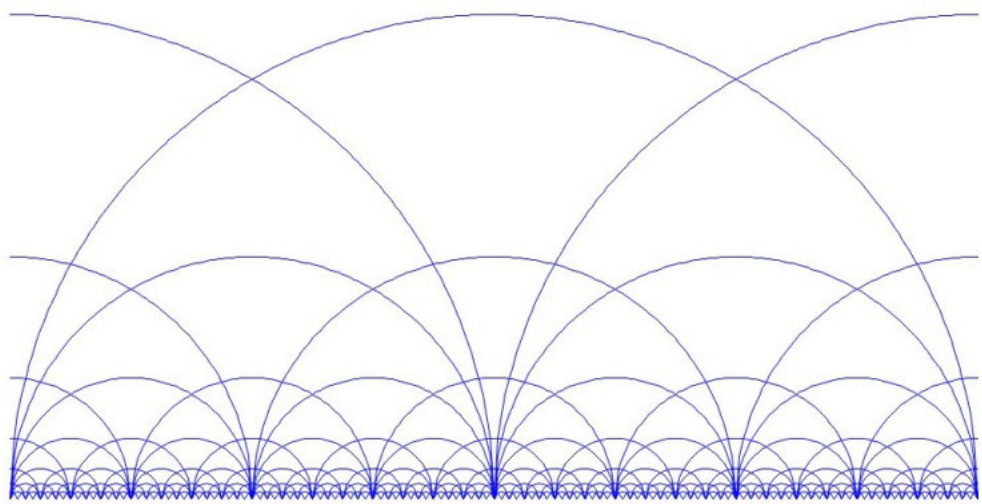


FIGURE 2. Geodesics in H, which we will mention in the future.