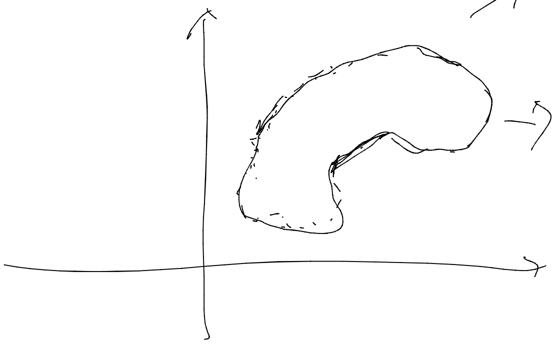


$\rightarrow R.$

① maximal open set $\subseteq R$

② minimal closed set $\supseteq R$ → closure.

<u>Def (interior)</u> $(X, \mathcal{T})$ ~~topological space~~. $A \subseteq X$, $\{U_i \mid i \in I\}$ be the set of all open sets ~~that~~ are contained in A. Then define,

$\mathring{A} := \bigcup_{i \in I} U_i$ as the interior of A.

<u>Properties</u>: $(X, \mathcal{T})$. $A \subseteq X$.

① $\mathring{A} \subseteq A$.

② $\mathring{A}$ is open.

③ A is open iff $A = \mathring{A}$

④ $U \subseteq A$ is open, then $U \subseteq \mathring{A}$.

<u>Examples</u>: ① $(\mathbb{R}$, half-open topology$)$ $[a, b)$ <u>interior $(0,1]$</u> ?

$\boxed{\text{interior}} \leftarrow \bigcup_{n=1}^{\infty} [\tfrac{1}{n}, 1) = (0,1)$

sketch: assume $1 \in \bigcup [a, b_i)$

$\boxed{ex?} \leftarrow$? $\exists N$, s.t. $1 \in [a, b_N)$ $[a, b_N) \not\subseteq (0,1]$

② $(\mathbb{R}$, finite-complement topology$)$ Interior of $(0,1]$?

interior of $(0,1] \Leftarrow R - \{P_1 \ldots P_n\} \not\subseteq (0,1]$

$= \phi$

$= \emptyset$

Def (closure) $A \subseteq (X, \tau)$, $\{V_j \mid j \in J\}$: collection of all closed subset of X that contains A. Then we define

$$\overline{A} := \bigcap_{j \in J} V_j$$ as the closure of A.

properties:

① $\overline{A} \supseteq A$

② $\overline{A}$ is closed

③ A is closed iff $A = \overline{A}$

④ $A \subseteq V$, V is closed, then $\overline{A} \subseteq V$

Example: Q1: ($\mathbb{R}$, usual topology) closure of $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\}$

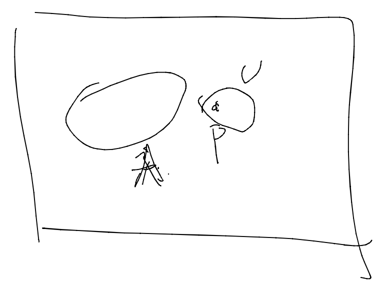
$$\overline{\{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}\}} = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots, \tfrac{1}{n}, \dots, 0\}$$

Lemma: $A \subseteq (X, \tau)$, then $p \in \overline{A}$ $\boxed{\text{iff}}$ for every open set $U \subseteq X$ containing p,

$$U \cap A \neq \emptyset.$$

$\Rightarrow$: $p \in \overline{A}$, and $\exists\, U \subseteq X$ open s.t. $p \in U$ and $U \cap A = \emptyset$.

Denote C by the complement of A.
$\subseteq$ $X \setminus A$



$p \in U \Rightarrow p \notin C$

C is closed and $A \subseteq C \Rightarrow \overline{A} \subseteq C$

$\Rightarrow p \in C$ contradiction. $\square$

and $A \subseteq C$ ...

$\Rightarrow p \in C$. contradiction. $\square$

$(-\infty, 0]$ $(1, \infty)$

Q: ($\mathbb{R}$, half-open topology) closure $(0,1]$?

two approaches: ① A^c ... closure of A ... $\left((-\infty,0) \cup (1,\infty)\right)^c$

$\left(\text{interior of } A^c\right)^c$

② $[0,1]$

Continuity:

Recall: HW $(X, d_1) \xrightarrow{f} (Y, d_2)$. f is continuous iff $\forall U \subseteq Y$ open, $f^{-1}(U) \subset X$ is open.

Def: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$ f is continuous if $\forall U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open.

Closed under composition: $(X, \tau_1) \underset{\text{continuous}}{\xrightarrow{f}} (Y, \tau_2) \underset{\text{continuous}}{\xrightarrow{g}} (Z, \tau_3)$

$g \circ f$ is continuous, because $\forall U \subseteq Z$ open, $g^{-1}(U)$ is open in Y, then $\underbrace{f^{-1} g^{-1}(U)}_{(g \circ f)^{-1}(U)}$ is open in X

Thm: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$. f is continuous iff

...

Thm: $(X, \tau_1) \longrightarrow (Y, \tau_2)$, f is continuous iff

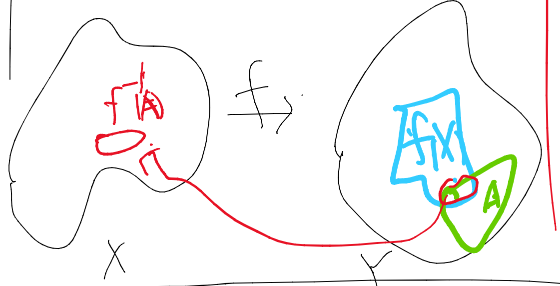
$\forall\, V \subseteq Y$ closed, $f^{-1}(V) \subseteq X$ is also closed.

Pf: Lemma: $\forall A \subseteq Y$, $f^{-1}(Y \setminus A) = X \setminus f^{-1}(A)$

$X = \{x \in X \mid f(x) \in A\}$

pf: $\{x \in X \mid f(x) \in Y \setminus A\} = \{x \in X \mid f(x) \notin A\}$ $\square$

f is not necessarily onto.



$f^{-1}(A) := \{x \in X \mid f(x) \in A\}$

$\Rightarrow$: $\forall\, U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open. $\Rightarrow$ $\forall C$ closed,

its complement $U = Y \setminus C$ is open, $f^{-1}(Y \setminus C)$ is open in X

$X \setminus f^{-1}(C)$ is open $\Rightarrow$ $f^{-1}(C)$ is closed. $\square$

$\Leftarrow$?

Def (open/closed maps) $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$

f is an open map if $\forall\, U \subseteq X$ open, $f(U)$ is open in Y

f is closed - - - $V \subseteq X$ closed, $f(V)$ — closed —

Example: $(\mathbb{R}, \text{half open}) \xrightarrow{f} (\mathbb{R}, \text{usual})$

$[a, b)$ $\quad$ $(\ )$ open set

open $\leftarrow f^{-1}$(open)

stronger $\quad$ weaker

open & closed

$x \longrightarrow x$

Q1: Is f continuous? Yes

Q2: Is f open? No

Q3: Is f closed? No

$[0,\infty)$

$[0,\infty)$ is neither open nor closed

$(0,1) = \bigcup [\frac{1}{n}, 1)$

Q3: Is f closed? ~~No~~ not closed

Homeomorphisms → equivalent relation between topological space.

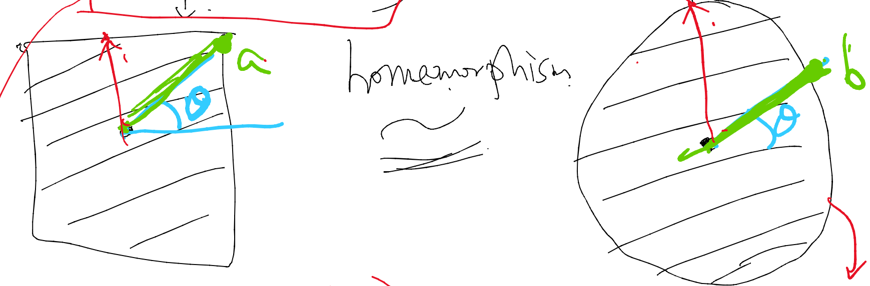
Def: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$ f is a homeomorphism, $\boxed{\text{inverse}}$

if f is bijective, continuous and $\boxed{\text{open}}$. $\boxed{f^{-1}}$ is continuous

If $\exists$ f homeomorphism between X, Y, then X, Y are homeomorphic.

$(\mathbb{R}^2, \text{Euclidean})$ $\boxed{[-1,1] \times [-1,1]}$ $\{x^2+y^2 \leq 1\}$

prove



ball in taxi-cab metric

ball in usual metric

$\boxed{\text{topologies are the same.}}$ $\Leftarrow$ $c\| \|_2 \leq \| \|_1 \leq C\| \|_2$

For each ray: f: a → b
end pt end pt.

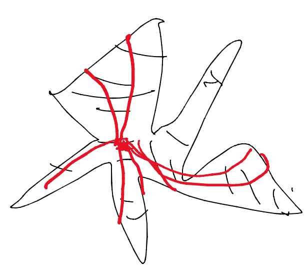
scale the interior of the ray.

(0, 0) → (0, 0)

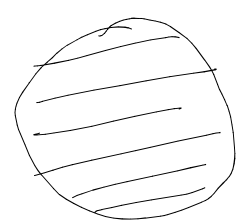
Same → Forward ray.

claim: f is homeo.

Q:



$\overset{?}{\cong}$ Yes



star shaped
induced from $(\mathbb{R}, \text{usual})$

S^1? No ← $\boxed{\text{harder}}$

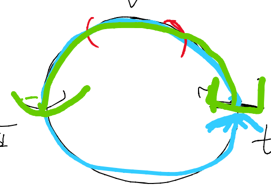
induced from $(\mathbb{R}, \text{usual})$

E.g.: $[0,1)$

$[a,b)$ open

$[0,c)$ open

$f: [0,1) \to (\cos 2\pi t, \sin 2\pi t)$



circle $\subseteq \mathbb{R}^2$

"open intervals"

induced topology from $\mathbb{R}^2$.

Q1: Is f bijective?

Yes

Q2: Is f continuous?

Yes!

Q3: Is f open?

No

$[0, \frac{1}{2})$ open $\to$ not open!