590 W3

Tuesday, January 26, 2021 10:35 PM

subspace and basis

idea: induced metric on a subspace from the original space

Def subspace topology / induced topology: (X, τ), $A \subseteq X$. We can topologize A by (A, τ_A).

$\tau_A = \{ U \cap A \mid U \text{ is open in } (X, \tau) \}$.

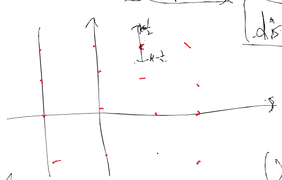
Eg 1: $[0,1) \subseteq (\mathbb{R}, \text{usual})$ — (a,b), $0 \le a < b \le 1$ and $[0, c)$ "generates" topology

2. $\mathbb{Z} \subseteq (\mathbb{R}, \text{usual})$ (lattice) what's the induced topology on $\mathbb{Z}$?

discrete ⟸ each point is open

3. $\mathbb{Z} \times \mathbb{Z} \subseteq (\mathbb{R}^2, \text{dictionary topology})$ induced topology on $\mathbb{Z} \times \mathbb{Z}$?

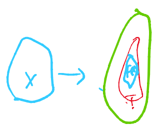
discrete



properties of subspace topology: Lemma: (X, τ_X), (Y, τ_Y); $\cup$ subspace topology $\cup$; (S, τ_S), (T, τ_T),

then:

1. If $S \in \tau_X$, then $\tau_S \subseteq \tau_X$.
2. $C \subseteq S$ is closed iff $\exists$ $A \subseteq X$ closed, st $C = A \cap S$.
3. $f: (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then $f|_S : (S, \tau_S) \to (Y, \tau_Y)$ is also continuous. $f|_S^{-1}(A) = f^{-1}(A) \cap S$



4. $f: (X, \tau_X) \to (Y, \tau_Y)$ is continuous, and $f(X) \subseteq T$, then $g: (X, \tau_X) \to (T, \tau_T)$ defined by $g(x) = f(x)$ is continuous.

Basis:

Idea: In a metric space, $\{B_\varepsilon(P)\}$ serves as a basis.

Def: (X, τ), $\beta \subseteq \tau$ is a basis of X if any $S \in \tau$ is a union of elements in β.

Eg. 1: $(\mathbb{R}, \text{usual})$, then $\{(a,b)\}_{a,b \in \mathbb{R}}$ is a basis.

2. $\{\mathbb{R}, \text{half-open}\}$ then $\{[a,b)\}$ is a basis. indeed we defined half open using such a basis!

3. $\{\mathbb{R}^2, \text{dictionary topology}\}$, $\{$all vertical line segments$\}$ is a basis.

Q: Is a basis necessarily minimal? $\exists \beta'$? $\beta' \subseteq \beta$

HW question

No! $\{\mathbb{R}, \text{usual}\}$ $\{(c,d) \mid c, d \in \mathbb{Q}\}$ is a basis.

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properties: Lemma 1: $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$, $\beta_X \to$ basis $\leftarrow \beta_Y$, f is continuous

iff $\forall B_Y \in \beta_Y$, $f^{-1}(B_Y)$ is open in X.

f is open iff $\forall B_X \in \beta_X$, $f(B_X)$ is open in Y.

more important

2. (X, τ), β is ... $W \subseteq X$ open?

T is "open" iff $\forall B_X \in \beta_X$, $T(B_X)$ is open " ".

(more important)

Lemma 2. (X, τ_X), β_X basis, $W \subseteq X$ (open?),

then $W \in \tau_X$ iff $\forall p \in W$ $\exists$ $p \in B_p \in \beta_X$, s.t. $B_p \subseteq W$.

Pf: $\Rightarrow$ (by def.) W is open, then $W = \bigcup_{i\in I} B_i$, then $\forall p$, $\exists B_p$. ✓

$\Leftarrow$ $\bigcup_{p\in W} B_p = W$ claim. $\bigcup_{p\in W} B_p \supseteq W$.

⇓ by def. so W is open. $B_p \subseteq W \Rightarrow \bigcup_{p\in W} B_p \subseteq W$. □

(X, τ), $\beta \subseteq \tau$ s.t. $\forall U \in \tau$, $U = \bigcup_{i\in I} B_i$, $B_i \in \beta$.

β is a basis of (X, τ).

Make new space from old: quotient space, product space.

equivalence relation: S set, $\sim$ relation is a equ relation if ($s_1 \sim s_2$, $s_1, s_2 \in S$)

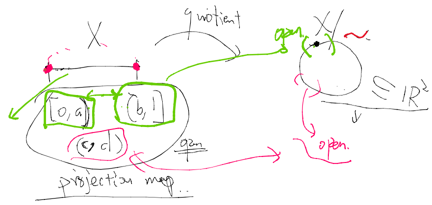
1. reflexive $x \sim x$
2. symmetric $x \sim y \Leftrightarrow y \sim x$
3. transitive $x \sim y$, $y \sim z$, then $x \sim z$.

Any. isomorphism is an equ relation.

For a set X with an equ relation $\sim$, $\forall a \in X$ we can define equivalence class, by $[a] := \{x \in X,\ x \sim a\}$,

define the quotient $X/\sim$ by $X/\sim := \{[x] \mid x \in X\}$.

Ex: $[0,1]/(0 \sim 1)$



Remark: $\pi: X \to X/\sim$, $x \mapsto [x]$

1. π is surjective,
2. π is injective iff $X = X/\sim$.

What's a good topology on $X/\sim$? A: π is continuous. We want the strongest topology.

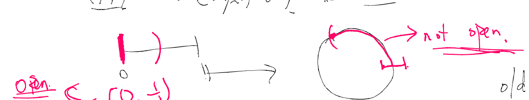
Def: (X, τ_X), $\sim$ is an equ. relation on X, we define

$$\tau_\sim := \{U \subseteq X/\sim \mid \pi^{-1}(U) \subseteq X \text{ open}\}.$$

$(X/\sim, \tau_\sim)$ is the quotient space of X by $\sim$ (w.r.t.).

Eg: $[0,1]/_{0\sim 1} \overset{\text{homeo}}{\sim} \bigcirc \subseteq \mathbb{R}^2$.

Remark: $(X, \tau) \xrightarrow{\pi} (X/\sim, \tau_\sim)$ is not necessarily open.



old space $[0,1]$ — X X circle

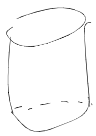
quotient ↙ new space $[0,1]/_{0\sim 1} \cong \bigcirc$

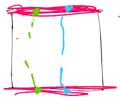
More example: $[0,1] \times [0,1] \subseteq \mathbb{R}^2 = (x, y)$

new space $[0,1]/_{0\sim1} \simeq$ ◯

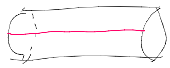
More example

1. cylinder

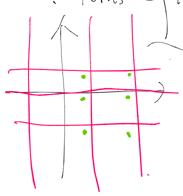


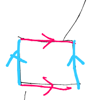
$X =$  $= [0,1]\times[0,1] \subseteq \mathbb{R}^2 = (x, y)$

$X/\sim$; $(x,y) \sim (\bar{x},\bar{y})$ iff $\begin{cases} x=\bar{x} \\ y=\bar{y} \end{cases}$ or $x=\bar{x}$, $y = 0$ and $\bar{y}=1$

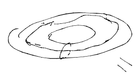
 $\subseteq \mathbb{R}^3$

2. Torus $= \mathbb{R}^2/\mathbb{Z}^2$ (lattice) $= \square/$ (sides and $\sim$) ; $X = \mathbb{R}^2$ ← usual





$X/\sim$; $(x,y) \sim (\bar{x},\bar{y})$ iff $x = p + \bar{x}$, $y = q + \bar{y}$, $p,q \in \mathbb{Z}$

$=$  $\subseteq \mathbb{R}^3$

$= S^1 \times S^1$

Remark: Fundamental domain : smallest region s.t. the gluing is well defined.

3. Projective space $\mathbb{RP}^2 = S^2/\sim$ (antipodal point) $\qquad \mathbb{RP}^n = S^n/\sim$

$X = S^2 := \{\vec{v} \in \mathbb{R}^3 \mid \|\vec{v}\| = 1\}$ (2 → n, 3 → n+1)

$X/\sim \qquad \vec{v} \sim \pm\vec{v}$

$Gr_{\mathbb{R}}(n, n+1)$ ← Grassmannian

fundamental domain

$D^2/\sim$ antipodal on ∂D^2

$D^2/\{\vec{v} \in \mathbb{R}^2 \mid \|\vec{v}\| = 1,\ \vec{v} = \pm\vec{v}\}$

$\mathbb{RP}^2$.

2nd approach for quotient space induced by a function (→ (π))

$X \xrightarrow{f} Y$ surjective, then we define "$\sim$" induced by f. (→ equivalence relation)

$\forall a, b \in X$. $a \sim b$ iff $f(a) = f(b)$.

$X/\sim$ quotient $\qquad X/\sim \to$ $f(p) \sim f(q)$ iff $p \sim q$

Def: (X, τ_X) $X \xrightarrow{f} Y$ surjective, we define the quotient topology

quotient $\tau_f := \{U \in Y \mid f^{-1}(U) \subseteq X \text{ open}\}$

strongest topology s.t. f is continuous.

f is the quotient map.

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product topology (X, τ_X) (Y, τ_Y) $X \times Y$ topologize; f_X, f_Y to X, Y

coarsest topology s.t. f_X, f_Y are continuous

More on basis, quotients, and products

Thm. (equivalence def of basis) X set, $\beta \subseteq 2^X$ s.t.

① $X = \bigcup_{B \in \beta} B$

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① $X = \bigcup_{B \in \beta} B$

② $B_1, B_2 \in \beta$, $\forall x \in B_1 \cap B_2$, $\exists\, B_x \subseteq B_1 \cap B_2$ and $B_x \in \beta$.

then $\tau = \{\text{Unions of elements in } \beta\} \cup \{\emptyset\}$ is a topology with basis β on X.

Pf: ⓐ Want X open, $\emptyset$ open ✓

ⓑ Want $U, V \in \tau$, $U \cap V \in \tau$.

By def of τ, $U = \bigcup_{i \in I} B_i$, $V = \bigcup_{j \in J} B_j$. Now $U \cap V = (\bigcup_{i \in I} B_i) \cap (\bigcup_{j \in J} B_j)$

$\forall x \in U \cap V$, by def of τ, $\exists\, i_x$ and j_x s.t. $x \in B_{i_x} \cap B_{j_x}$

Then: $\exists\, B_x \in \beta$, s.t. $B_x \subseteq B_{i_x}$ and $B_x \subseteq B_{j_x}$

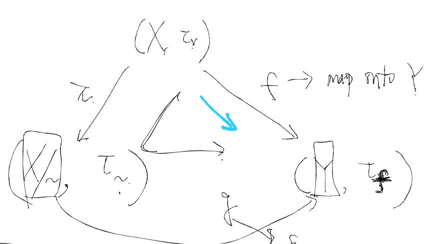
claim: $U \cap V = \bigcup_{x \in U \cap V} B_x$ — How to prove: $\supseteq$, $\subseteq$

ⓒ $\forall k \in K$ index set, $A_k \in \tau$ open.

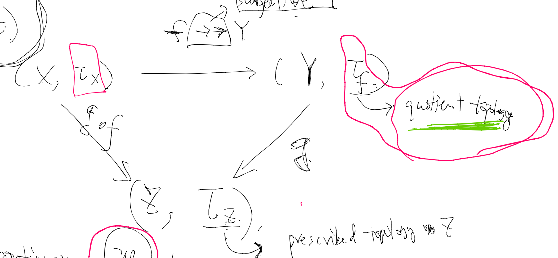
Want $\bigcup_{k \in K} A_k \in \tau$. By def of τ, $A_k = \bigcup_{i \in I} B_i$

$\bigcup_{k \in K} A_k = \bigcup_{k \in K} (\bigcup_{i \in I} B_i) = \bigcup$ elements in β, is open. □

Recall. Two defs of quotient spaces



Lemma (Important)

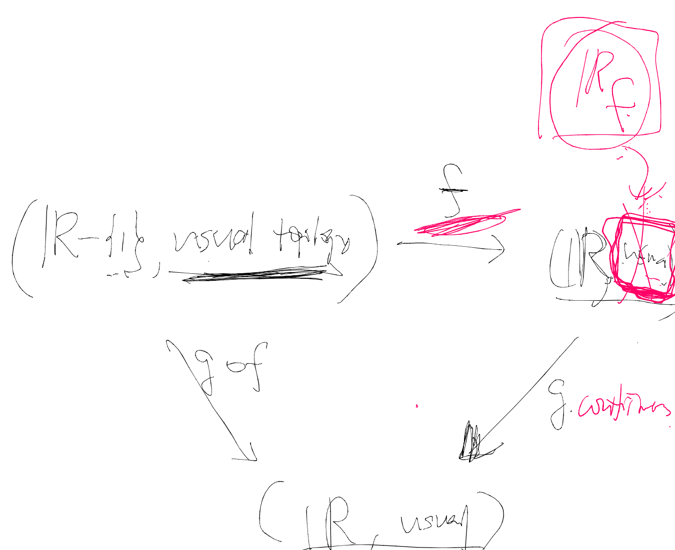


claim: g is continuous $\iff$ $g \circ f$ is continuous

Pf: $\Rightarrow$ easy: g is continuous, f is the quotient map, hence continuous. $g \circ f$ is continuous.

$\Leftarrow$: Know $g \circ f$ continuous $\iff$ $\forall A \in \tau_Z$, $(g \circ f)^{-1}(A) \in \tau_X$. Know $\forall B \in \tau_f$, $f^{-1}(B) \in \tau_X$.

want $\forall A \in \tau_Z$, $g^{-1}(A) \in \tau_f$ ⟵ $\forall A \in \tau_Z$, $f^{-1}(g^{-1}(A)) \in \tau_X$.



claim: ① $f := \begin{cases} x^2, & \forall x < 1 \\ \frac{1}{x-1}, & \forall x > 1 \end{cases}$ — f is surjective, f is continuous

② $g := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ — not continuous; $(\mathbb{R}$, usual$) \to (\mathbb{R}$, usual$)$

③ $g \circ f = \begin{cases} 1 & x < 1 \\ -1 & x > 1 \end{cases}$ — continuous

Thm (Functoriality of quotient maps)

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$g \circ f := \{ \ldots \quad x>1$ — Continuous

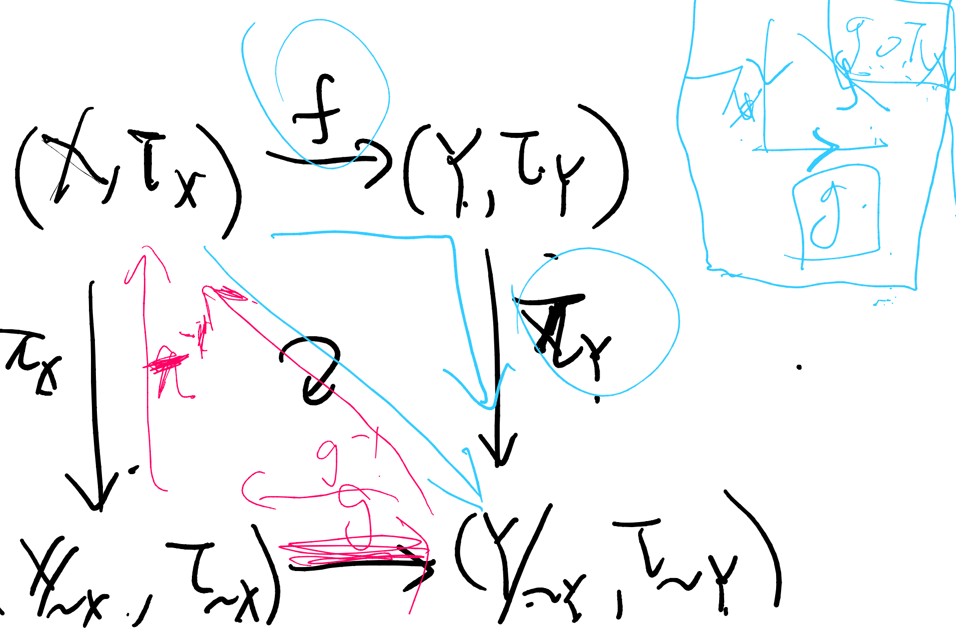
$(X, \tau_X) \xrightarrow[f]{\text{homeo}} (Y, \tau_Y)$

$\sim_X \overset{\text{"same"}}{\longleftrightarrow} \sim_Y$

$x_1 \sim x_2$ iff $f(x_1) \sim f(x_2)$

claim: $(X/\sim_X, \tau_{\sim_X}) \overset{\text{homeo}}{\simeq} (Y/\sim_Y, \tau_{\sim_Y})$

Pf: un



$\exists\, g: X/\sim_X \to Y/\sim_Y$

g is a homeo, and ...

g continuous

g^{-1} continuous

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product spaces:

weakest topology s.t. $\pi_X: X \times Y \to X$

How? $\neq$ {product of open sets}

first ...