Properties of product spaces

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\pi_X : (x, y) \to x$

$\pi_Y : (x, y) \to y$

Facts ① $(X, Y, \tau_{X\times Y}) \xrightarrow{\pi_X} (X, \tau_X)$

then π_X and π_Y are continuous

② Same setting, then

π_X and π_Y are open!

quotiant maps are not always open!!

Lemma: π_X and π_Y are open. products of open $\in X$ and Y

pf: π_X sends base open sets to open sets,

$B_i \in \mathcal{B}_{X\times Y}$

$$\pi_X\left(\bigcup_i B_i\right) = \bigcup_i \pi_X(B_i) \text{ open. } \square$$

Lemma (Important)

$$\begin{tikzcd} & (X\times Y, \tau_{X\times Y}) \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ (X, \tau_X) & & (Y, \tau_Y) \\ & (A, \tau_A) \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, bend right, "h"'] & \end{tikzcd}$$

f and g are continuous

$\overset{\text{iff}}{\Longleftrightarrow}$ h is continuous.

Note: $\boxed{\begin{aligned} f &= \pi_X \circ h \\ g &= \pi_Y \circ h \end{aligned}}$

pf: $\Leftarrow$ done, because π_X, π_Y are continuous.

$\Rightarrow$ We assume f, g are continuous,

Want: $h^{-1}(\text{every base open sets}) \to$ open in A

$U \times V$; $U \in \tau_X$ and $V \in \tau_Y$.

$f^{-1}(U)$ and $g^{-1}(V) \in \tau_A$

$h^{-1}(\prod_{i\in 2} U_i)$

$$\begin{aligned}
\boxed{h^{-1}(U\times V)} &= \{ a \in A \mid h(a) \in U \times V \} \\
&= \{ a \in A \mid (f(a), g(a)) \in U \times V \} \\
&= \{ a \in A \mid f(a) \in U \text{ and } g(a) \in V \} \\
&= f^{-1}(U) \cap g^{-1}(V)
\end{aligned}$$

$\to$ intersection of opensets in A

$\mathbb{R} \to \mathbb{R}$

$f_1 = x^2$ continuous

$f_2 = x^3 + x$

$f_3 = \sin x$

Jun's Notebook » Quick Notes

Shapes   Ink to Shape   Ink to Text   Ruler

$f_3 = \sin x$

$\mathbb{R} \to \mathbb{R}^3 : x \mapsto (x^2, x^2+x, \sin x)$ is continuous

$= f^{-1}(U) \cap g^{-1}(V)$ → intersection of open sets in A → open in A! □

$\bigcap_{i \in I} \pi_i^{-1}(U_i)$ → not necessarily open

Infinite product:

$\prod_{i \in I} X_i$ I is an index set.

$\{0,1\}^{\mathbb{R}}$

choice of topology:

box topology $\boxed{\tau_\square}$: has a basis $\beta_\square$ {product of open sets in X_i's}

'Important' lemma still holds? No!

we have to deal with ∞ $\cap$.

Correct choice: product topology $\boxed{\tau_\pi}$ for ∞-components:

$\boxed{\beta_\pi}$ → basis = { product of finite open sets in X_i's }.

Important lemma still holds.

Compare $\beta_\square$ and β_π.

fixed value

$(X_1 \times X_2) \subseteq (\prod_{i \in I} X_i)$

subspace topology of $\beta_\square$ or β_π

① $\beta_\square$ and β_π are both generalizations of $\beta_{X \times Y}$.

Same. → product topology on $(X_1 \times X_2)$

② $\beta_\pi \subseteq \beta_\square$ "weakest" topology on product spaces $\boxed{\beta_\pi}$.

Compactness   Hausdorff.

Rmk: quotient space $\overline{D_2} \cup_f M$ quotient $\overline{D_2} \amalg M$, f

How to distinguish spaces?

- Prove $X \overset{\text{homeo}}{\cong} Y$: we only need a map $X \xrightarrow{f} Y$ → homeo.
- Prove $X \not\cong Y$, hard. ← invariant property under homeo

  Cpt, Hausdorff, connectedness, $|X|$.

Cpt → finiteness

Def. (X, τ_X), $A \subseteq X$. A is compact if

$\forall$ collection $C \subseteq \tau_X$ if $A \subseteq \bigcup_{i \in C} U_i$, we can find a finite $F \subseteq C$ s.t.

$$A \subseteq \bigcup_{i \in F} U_i.$$

Thm 1: Any closed subset A of a compact space X is compact.

Pf: $X = A \cup (X \setminus A)$ □

$\overline{D_n}$, S_n are compact.

Thm 2: (Heine-Borel): $(\mathbb{R}^n, \text{usual})$, $A \subseteq \mathbb{R}^n$ is cpt iff A is closed and bounded.

Thm 3: (Tychonoff) (X, τ_X), (Y, τ_Y) are cpt space, then $(X \times Y, \tau_{X \times Y})$ is cpt. finite

Hausdorff is a separation property → T_0 T_1 $T_{1\frac{1}{2}}$ T_2 T_3 T_4

Def (Hausdorff): (X, τ_X) $\forall\, x, y \in X$

$\exists$ U and $V \in \tau_X$ s.t. $U \cap V = \phi$ and $x \in U \subseteq X$, $y \in V \subseteq X$.

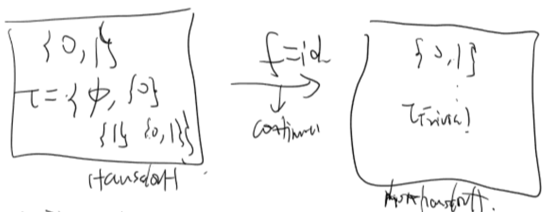
Hausdorff.

Any reasonable object should be hausdorff

Eg. Non-Hausdorff space: $\{0,1\} = X$ $\quad \tau_X = \{\phi, X\}$. We can not seperate $0, 1$.

Any set S with $|S| \geq 2$, τ_{Trivial}, $\to$ non-Hausdorff.

Eg: (continous map does not preserve T_2):



Thm: $(X, \tau_X) \xrightarrow{g} (Y, \tau_Y)$, g is a homeomorphism, then X is Hausdorff iff Y is hausdorff.

pf: $\square$

Interaction between cpt and Hausdorffness.

Thm (X, τ_X) Hausdorff, $A \subseteq X$ cpt., A is closed.

pf: Next week $\square$

Rmk: (X, τ_X) is cpt and Hausdorff. $A \subseteq X$ closed $\Longleftrightarrow$ cpt.

Application: $\underset{D_n}{(X, \tau_X)} \xrightarrow{f} \underset{S_n}{(Y, \tau_Y)}$ continuous surjection, X and Y are cpt and Hausdorff, then f is a quotient map.

Pf: Want: ① $f^{-1}(U)$ is open in X iff U is open in Y $\leftarrow$ f continuous

quotient $\leftarrow$ ② ... $\leftarrow$ weakest quotient topology ← ② $\mathcal{T}_Y = \{ U \subseteq Y \mid f^{-1}(U) \text{ is open} \}$ ← weakest topology

$X \xrightarrow{f} Y$

$V \subseteq Y$ $f(f^{-1}(V)) = V$ because f is surjective.

closed closed

want f is a close map

$Z = f^{-1}(V)$ closed in X $\xRightarrow{X \text{ is cpt}}$ Z is cpt in X

$\xRightarrow{f \text{ is continuous}}$ $f(Z)$ is cpt in Y $\xRightarrow{Y \text{ is Hausdorff}}$ V is closed.

□

This proves ②.

$\boxed{\text{continuous}}$

$|\vec{x}| \le 1 \in \overline{D_n} \longrightarrow \mathbb{R}^{n+1}$

$x \longmapsto \left(\frac{2\sqrt{\|x\|(1-\|x\|)}}{\|x\|}, 2\|x\|-1 \right)$ Image is S_n.

$\overline{D_2} \cup_f M$ ← $\boxed{\text{polygon representation}}$

$\| \longrightarrow$ How to prove?

$\mathbb{RP}^2$

$\|$

$\overline{D_2}$