590 W5

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Connectedness & path connectedness

Def: A space is connected if it can not be seperated by two open (sub)sets

Eg. $(\mathbb{R}, \text{usual})$ is connected, (a,b) are connected subsets.

Rmk: connected domains have connected image under continuous maps.

Def (connected components) (maximal!) (X, τ_X), $p \in X$, take all connected subsets containing p, $\{C_i,\ i \in I \mid C_i \overset{\subseteq X}{\text{connected}},\ p \in C_i\}$

then $C_p := \bigcup_{i \in I} C_i$ is the connected component containing p.

Lemma: (X, τ_X), C_p as above, then ① C_p is connected

② C_p and C_q are connected components, then either $C_p = C_q$ or $C_p \cap C_q = \emptyset$

Pf: ① = Flower lemma → $X = \bigcup_{j \in J} Y_j$, Y_j are connected, and $Y_a \cap Y_b \neq \emptyset$ $\forall a, b \in J$, then X is connected

p

□

② Assume $\boxed{C_p \cap C_q \neq \emptyset}$, want $C_p \overset{\text{(initial)}}{=} C_q$

pf. $C_p \cup C_q \ni C_p \cup C_q = \{C_k, k \mid C_k \text{ connected},\ p \in C_k\} \overset{\text{by def}}{=} C_p$

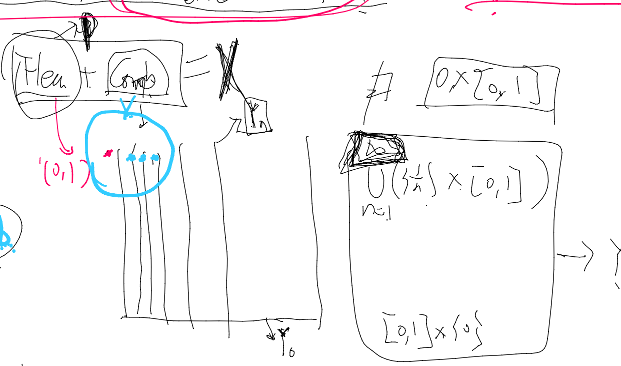
$\Rightarrow C_q \subseteq C_p$ and vice versa.

□

Topologist's sin curve

Eg: connected space that doesn't look like connected → path connected

$(\mathbb{R}^2, \text{usual}) \to$ subspace $=$ (Flea + Comb) $= X$ $\nexists$ $[0 \times [0,1]]$



"We cannot travel" from $(0,1)$ to the comb

But the space X is connected (X, τ_X) subspace topology from $\mathbb{R}^2$

pf: Firstly, Y is connected ← by induction, Flower lemma

claim $Y_0 \cup Y_1$ is connected, because Y_0, Y_1 are connected, $Y_0 \cap Y_1 = (1, 0)$.

$\bigcup_{k=0}^{n} Y_k \cup Y_{n+1}$ is connected if $\bigcup_{k=0}^{n} Y_k$ is connected.

Non-separation of p and Y. If X and U s.t. U, V open, and $U \cap V = \emptyset$, $U \cup V = X$,

then if $p \in V$. WLOG we can assume $Y \in U$.

and. $U \cap V = \phi$, $U \cup V = X$,

then if $p \in V$, WLOG we can assume. $Y \subseteq U$

(X, τ_X) is the subspace topology, $\exists$. $B_p(\varepsilon) \subseteq V$ s.t. $B_p(\varepsilon) \cap Y = \phi$,

This is contradiction. because $\exists N > \frac{1}{\varepsilon}$ s.t. $B_p(\varepsilon) \cap Y_N \neq \phi$.

Let's introduce path connectedness "more intuitive".

Def (path): $[0,1] \xrightarrow{f} (X, \tau_X)$ continuous map is called a path in X.

Def (path connected): $p, q \in (X, \tau_X)$, p is path connected to q if $\exists f(t)$ path s.t. $f(0) = p$ and $f(1) = q$.

$A \subseteq X$ is path connected if $\forall p, q \in A$, p, q are ~~connected~~ path connected,

Def (path connected component): (X, τ_X), $p \in X$, $C_p := \bigcup_{i \in I} C_i$ where C_i are $\{ C_i \subseteq X, i \in I \mid p \in C_i, C_i \text{ are path connected} \}$

Lemma: ① C_p path connected,

② C_p, C_q path connected component, then either $C_p \cap C_q = \phi$ or $C_p = C_q$.

pf: $\square$

Rmk: ① intuitive.

② "Connected" is a $\boxed{\text{negative}}$ def, but "P. connected" is positive.

Eg: Flea + Comb is connected but not path connected → "harder"

$\boxed{\text{Thm I: Path connected} \Rightarrow \text{Connected.}}$ $\boxed{\text{pf next time}}$

Lemma 2: (Pasting or flower lemma): $\boxed{①}$ If $f(t)$ and $g(t)$ are continuous paths st $f(1) = g(0)$, then the concatenation path $(f * g) := \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$ is also a ~~connected~~ continuous path.

⇓

② $X = \bigcup_{i \in I} Y_i$, Y_i are path connected and $Y_a \cap Y_b \neq \phi$ then X is path connected.

Categorical language for topological spaces

Def. Category: C consists of.

- ① a class (not necessarily a set) of objects $Ob(C)$
- ② a class of morphisms (arrow) $Hom(C)$

Eg. **Set**: category of sets

Vect: —— Linear space

Gp: - - - group

Mod: - - - - module

not necessarily a set

③ a class of morphisms (arrow) $\mathrm{Hom}(C)$

③ domain object. $\mathrm{Hom}(C) \to \mathrm{Ob}(C)$

④ codomain object. $\mathrm{Hom}(C) \to \mathrm{Ob}(C)$

⑤ $\forall\ a \xrightarrow{f} b \xrightarrow{g} c$, $\exists\ g \circ f$ → composition of morphism exists

Gp: . . . . group

Mod: - - - - module

$\boxed{\mathrm{Top}}$: - - - - topological spaces

Also. a) associativity of morphisms $(f \circ g) \circ h = f \circ (g \circ h)$

b) $\exists$ of id morphism, $\forall x \in \mathrm{Ob}(C)$, $\exists\ \mathrm{Id}_x : x \to x$ s.t. $f = f \circ \mathrm{Id}_x$, $g = \mathrm{Id}_x \circ g$

Id_x x f and g a b

Pros.: objects are "less" important; $\boxed{\text{morphisms}}$ are essential.

arrow language about morphisms

Rk 1: $\mathrm{Ob}(C)$ is a set, then C is a $\boxed{\text{small}}$ category.

rk 2: $\boxed{\text{Functors}}$ between categories.

$$\begin{array}{ccc} \mathrm{Top} & & (X, \tau_X) \xrightarrow{f} (Y, \tau_Y) \\ \downarrow F & \text{forgetful functor} & \downarrow F \qquad\qquad \downarrow F \\ \mathrm{Set} & & X \xrightarrow{F(f)} Y \end{array}$$

Rk 3: "dual" category or "dual" construction in a category

domain $X \xrightarrow{f} Y$

dual $X \xrightarrow{f} Y \to$ codomain

Universal constructions (limit/colimit) in Top (or in C)

Def. (initial obj.) (final obj.) in Top

$\{(X_i, \tau_i) \in \mathrm{Top}\}_{i \in I}$ is a class of objects. set

S is a set

(Initial obj.) $\{S \xrightarrow{f_i} X_i\}$ be a set of functions, then the initial topology $\tau_{\mathrm{ini}}(\{f_i\})$ is the minimal collection of opensets in S. s.t. f_i's are continuous.

→ limit. → product space

(final obj) $\{X_i \xrightarrow{g_i} S\}$ - - - - - - - final

colimit. - - $\tau_{\mathrm{final}}(\{g_i\})$. - - - maximal - - - - -

- - - g_i's continuous.

coproduct || disjoint union

Ex: - (-) - (F) - - (-) -

coproduct = disjoint union

g_i := continuous,

Eg. 1: $\triangleleft\ S \subseteq (X, \mathcal{T}_X)$ $S \xrightarrow[\text{inclusion}]{f} X$, then $(S, \mathcal{T}_{ini}(\{f\}))$ is the subspace topology

$\triangleleft\ (X, \mathcal{T}_X) \xrightarrow[\boxed{\text{surjective}}]{g_i} Y$, then $(Y, \mathcal{T}_{final}(\{g_i\}))$ is the quotient topology

Q: What if S arb set, $(X_i, \mathcal{T}_i) = \mathrm{Obj}(\mathcal{C})$, what. $(S, \mathcal{T}_{ini}\{f_i\})$ → discrete.

S pt. $(X, \mathcal{T}_i) = \mathrm{Obj}(\mathcal{C})$ $(S, \mathcal{T}_{fin}\{f_i\})$ → trivial

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Top formal diagram

Category $\mathcal{C}$, Category of diagrams of shape J

$F: J \to \mathcal{C}$ a functor,

A cone $\boxed{N}$ to F is an object N, together with a family of morphisms $\phi_X : N \to F(X)$

$$\begin{tikzcd} & N \arrow[dl, "\phi_X"'] \arrow[dr, "\phi_Y"] & \\ F(X) \arrow[rr, "F(f)"] & & F(Y) \end{tikzcd}$$

s.t. $\forall$ morphism $F(f): F(X) \to F(Y)$ diagram commutes, i.e. $\phi_Y = F(f) \circ \phi_X$

A limit of $F: J \to \mathcal{C}$ is a cone (L, ϕ) s.t. $\forall$ cone N, $\exists!$ morphism $\boxed{N \xrightarrow{u} L}$ s.t. diagram commutes

$$\begin{tikzcd} & N \arrow[d, "u"] \arrow[ddl, bend right, "\psi_X"'] \arrow[ddr, bend left, "\psi_Y"] & \\ & L \arrow[dl, "\phi_X"] \arrow[dr, "\phi_Y"'] & \\ F_X \arrow[rr, "Ff"] & & F_Y \end{tikzcd}$$

Eg. in Top. $J = \{\cdot \ \cdot\}$

$$\begin{tikzcd} & X \times Y \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

what is a good obj L, s.t.

$$\begin{tikzcd} & N \arrow[d, "u"] \arrow[ddl, "\psi_X"'] \arrow[ddr, "\psi_Y"] & \\ & L \arrow[dl, "\phi_X"] \arrow[dr, "\phi_Y"'] & \\ X & & Y \end{tikzcd}$$

commute?

In $\boxed{\text{Set}}$ $L = X \times Y$

In Top $(X, \mathcal{T}_X) \leftarrow (X \times Y, \mathcal{T}_{X \times Y}) \rightarrow (Y, \mathcal{T}_Y)$ weakest

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Universal property of limit / colimit

limit → product; colimit → quotient

"Important Lemmas"

product topology generated by finite product

$$\begin{tikzcd} \prod X \arrow[d, "\pi_i"] & \\ X_i & Z \arrow[l, "f \circ \pi_i"] \arrow[ul, "f"'] \end{tikzcd}$$

f is continuous iff all $f \circ \pi_i$'s are continuous

$$\begin{tikzcd} X \arrow[d, "q"] \arrow[dr, "f \circ q"] & \\ Y \arrow[r, "f"] & Z \end{tikzcd}$$

colimit

$\forall\ Z \in$ Top, f is continuous iff $f \circ q$ is continuous

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Top is not the mostly used category, instead

Compactly generated Hausdorff top. ← CG Haus is

$(X, \mathcal{T}_X)$ is cpt. gen.

(X, τ_X) - is cpt gen.

if. Any subspace $A \subseteq X$ is closed iff $A \cap K$ is closed in K, $\forall K \subseteq X$ cpt