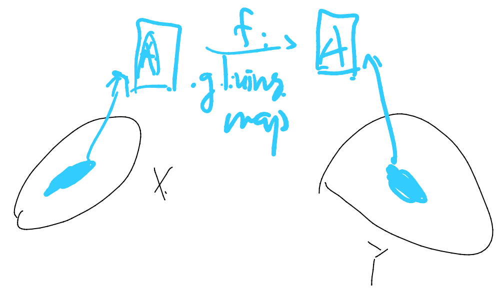
590 W6

Friday, March 5, 2021   11:59 AM

Recall



$\longmapsto$ construction of $\boxed{\text{surfaces}}$

$X \cup_A Y = X \amalg Y / x \sim f(x)$ $\longrightarrow$ special case of quotient space.

Def (topological surface) Σ is a hausdorff topological space, s.t. $\forall x \in \Sigma$, $\exists$ U_x open, s.t. U_x is homeomorphic to an open subset of $\mathbb{R}^2$.

Def (topological manifold of dim $= n$) M^n is a (paracompact) $\boxed{\text{hausdorff}}$ space, s.t. it is locally homeomorphic to $\mathbb{R}^n$

$\forall$ subset A, open subsets

Q1: What is paracompact?

Compact: every open covering has a finite refinement

Paracompact: every open covering has a locally finite refinement

$\forall x \in A$, $\exists$ U_x open s.t. U_x has a finite subcover.

why paracompact?

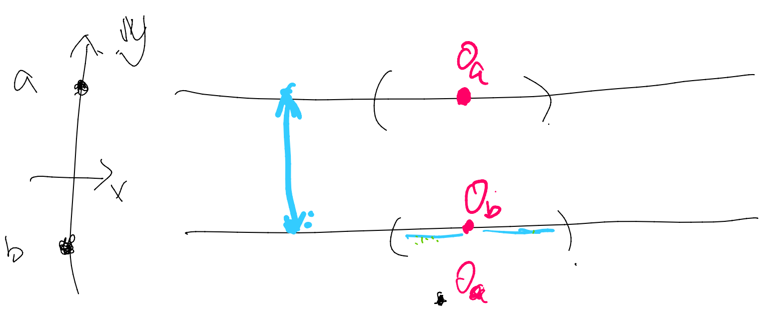
metrizable $\Rightarrow$ paracompact

paracompact + Hausdorff + locally $\mathbb{R}^n$ $\Rightarrow$ metrizable.

Q2: locally homeo to $\mathbb{R}^n$ $\overset{?}{\Rightarrow}$ Hausdorff? No!

metrizable $\Rightarrow$ Hausdorff

counterexample (double eyed $\mathbb{R}^1$)

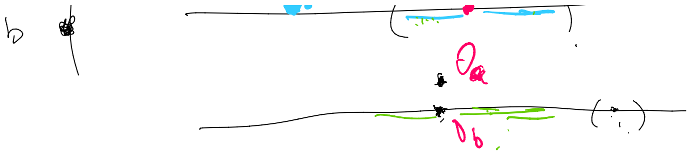


$(\mathbb{R}^1$, usual$)$

glue: $\boxed{\forall x \neq 0,\ (x,a) \sim (x,b)}$

$(\mathbb{R}^1$, usual$)$

$\mathbb{R}^1 \times a \amalg \mathbb{R}^1 \times b / \sim$



$\mathbb{R}^1 \times a \amalg \mathbb{R}^1 \times b \,/\, (x,a) \sim (x,b) \quad \forall x \neq 0$

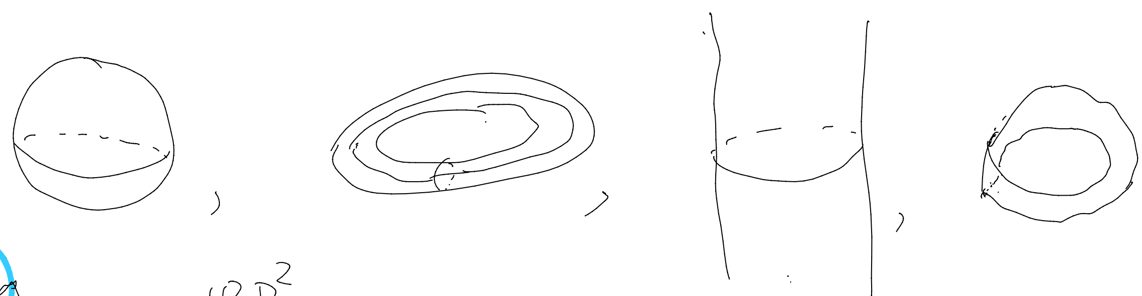
① locally $\mathbb{R}^1$ ✓ $\forall x \neq 0$ ✓

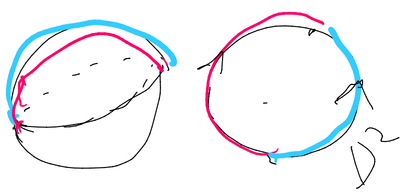
O_a ~~and~~ or O_b — $\boxed{(-\varepsilon, 0) \cup (0, \varepsilon) \cup \{O_a\}}$ or O_b → open → $(-\varepsilon, \varepsilon) \subseteq \mathbb{R}^1$

② not Hausdorff := cannot seperate O_a and O_b

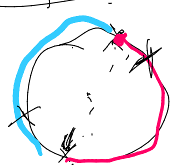
$\forall\, U_{O_a}$ and $\forall\, U_{O_b}$ nbhd of O_a and O_b, they always intersect.

Examples:



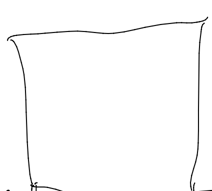


$\mathbb{RP}^2$



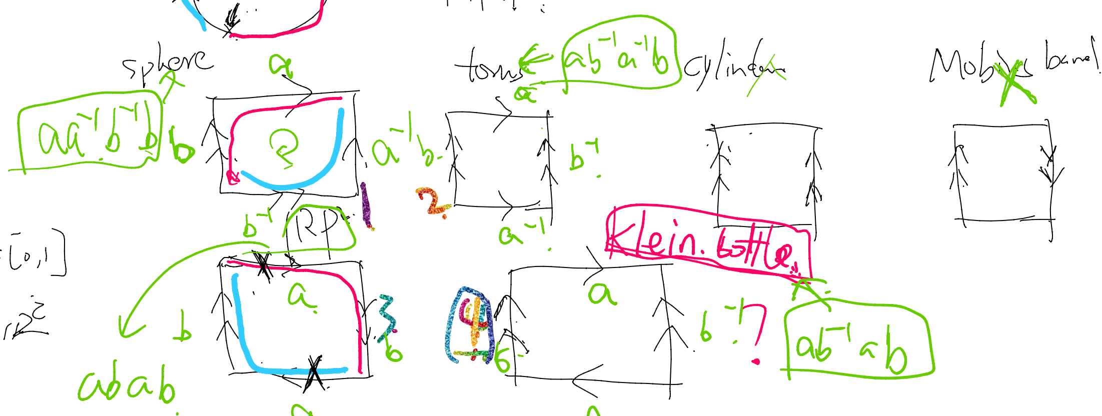
$\overline{D_2} / \vec{x} = -\vec{x},\ |\vec{x}| = 1$

construction from polygons



$= [0,1] \times [0,1] \subseteq \mathbb{R}^2$

(polygon, word) → surface



write down all letters as a word

- ◁ clock wise order.
- ◁ different edges ⟷ different letters
  - edges glued together ⟷ same letter
- ◁ a and a^{-1} stands for $\boxed{\text{different orientations}}$ of the same edge.
  - go clockwise along the polygon, if this match the arrow, a

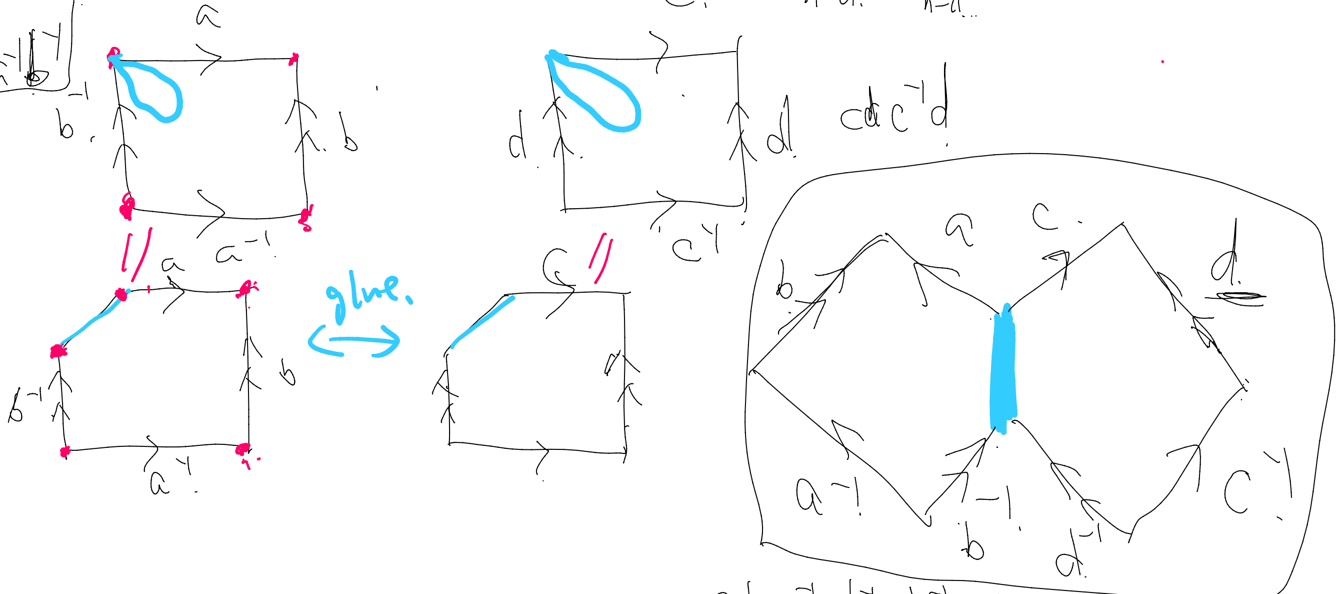
go clockwise along the polygon, if this match the arrow, a. otherwise a^{-1}

Construction of more surfaces:

$T^2 \setminus D^2$ (open) $\qquad T^2 \setminus D^2$

$\underbrace{T^2 \# T^2}_{A \quad B} \to$ connect sum of $\underset{n\text{-d}}{A^n}$ and $\underset{n\text{-d}}{B^n}$ = ① remove open D^n from A and B ② glue $\partial D^n = S^{n-1}$ together

$\boxed{ab\,a^{-1}b^{-1}}$ $\qquad cdc^{-1}d^{-1}$



$cdc^{-1}d^{-1}b^{-1}a^{-1}ba$

rename

$\Sigma_2 \leftarrow a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$

More generally: $\#_g T^2 = \underbrace{T^2 \# \cdots \# T^2}_{g} \to$ genus g surface

$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

g $\qquad$ g $\qquad a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$

Thm (1): Any compact $\boxed{\text{orientable}}$ surface is homeomorphic to either $S^2 = \Sigma_0$ or Σ_g, $g \geq 1$.

Lemma: $\boxed{\mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2 = T^2 \# \mathbb{RP}^2} \to$ HW problem

Thm (2) — — non-orientable — — — — to $\#_n \mathbb{RP}^2$'s

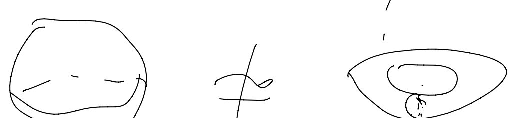
Topological invariant: orientability, Euler number.

essential tool to distinguish topological spaces

homotopy, homology groups $\downarrow$ easier direction

$X \xrightarrow{f} Y$: $\exists f$ s.t. f, f^{-1} continuous $\Rightarrow X \overset{\text{homeo}}{\cong} Y$

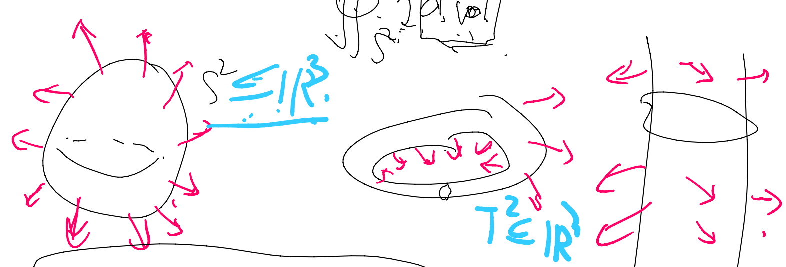
hard. $\leftarrow$ $\boxed{\nexists}$ f s.t. f, f^{-1} continuous $\Rightarrow X \ncong Y$



$\boxed{\chi(S^2) = 2 \neq 0 = \chi(T^2)}$

Orientability of surfaces:

Recall Stokes' theorem $\qquad \iint_{S^2} \ldots \boxed{d\,\mathrm{vol}}$



Def 1: If $\exists$ a consistent choice of $\boxed{\text{normal vector field}}$, then the surface is orientable.

(extrinsic definition)

not orientable

Def 2. (orientability): If on Σ, $\nexists$ open subset that is homeomorphic to the (open) Mobius band, then Σ is orientable. Otherwise Σ is not orientable.

our standard def.

RK 1: this way defined orientability is a topological invariant.

RK 2: $\mathbb{RP}^2$, by this definition, is not orientable. $D^2 \cup_{S^1} \boxed{M}$ $\boxed{\exists M \subseteq \mathbb{RP}^2}$ $[0,1]\to$

orientable

Def. 3 (homotopy) If on Σ, any continuous loop $f: [0,1] \to \Sigma$ s.t. $f(0) = f(1)$

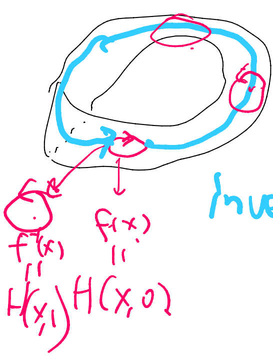
is not homotopic to its inverse, then Σ is orientable

$\Vert$

can not be continuously deformed to

$H(x, \frac{2}{3})$

$H(x, \frac{1}{2})$



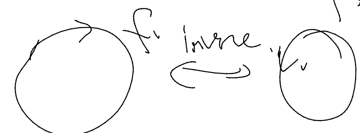
$f(x)$ $f(x)$

$\Vert$ $\Vert$

$H(x,1)$ $H(x,0)$

Inverse of a path or a loop

f^{-1}



f, inverse

$f(x): [0,1] \to \Sigma$

$f(1)$

$f(0)$ f

$f^{-1}(x) := f(1-x)$

$f^{-1}(1)$ f^{-1} $f^{-1}(0)$

homotopy of f and f^{-1} is a continuous function $H(x,t): [0,1] \times [0,1] \to \Sigma$

$\Vert$

continuous deformation

s.t. $H(x,0) = f(x)$

$H(x,1) = f^{-1}(x)$

How to prove 3 def's are equivalent $\longmapsto$ not easy

Euler number (Euler characteristic)

S^2

homeo

$=$

4

$V - E + F$

vertices, # edges, # faces

$4 - 6 + 4 = 2$

surface

1 pt.

4   surface,   1 pt. $F - 6 + 4 = 2$

$1 - 2 + 1 = 0$

Def (subdivision) of a compact surface Σ is a partition of Σ into

1) vertices (finite points on Σ)
2) edges (finitely many disjoint subsets on Σ s.t. each being homeo to $(0,1) \subset \mathbb{R}$)
3) faces (finite disjoint subsets on Σ ... homeo to open D_2)

$\{x^2+y^2<1\} = \mathbb{R}^2$

s.t.

a) faces are connected components of $\Sigma \setminus \{$vertices and edges$\}$
b) no edges contain a vertex.
c) "each edge begins and ends with a vertex" → same or different

If e is an edge, $\exists\, v_1, v_2$ vertices s.t. $\exists\, f: [0,1] \to \Sigma$ s.t. $f(0) = v_1$, $f(1) = v_2$, $f((0,1)) = e$.

Def (Euler number) Σ with a subdivision, $\chi(\Sigma)$ is called the Euler number, defined as $V - E + F$.

Q: Does Euler number depend on the choice of subdivision?

Thm (Nigel Hitchin): Euler number is a topological invariant, independent of subdivision.

"Pf"

For any continuous path $f: [0,1] \to \Sigma$ we define the ~~boundary~~ differential map ∂f

$f : [0,1] \to \Sigma$, we define the $\boxed{\text{boundary map } \partial f}$ to be the formal linear comb $f(0) + f(1)$

another def of Euler number using homology.

simplicial homology
||
singular homology

$\boxed{C_0}$: linear space given by formal linear comb of points with $\mathbb{Z}_2$ coefficients → additive.

$0X + 1X + 0X + X$

∂_1

C_1: linear space. paths

C_2: linear space. $\boxed{\text{triangles}}$ with $\mathbb{Z}_2$ coefficient

for a continuous map

we define $\partial_2 g :=$ linear combination of edges $AB + BC + AC$

poincare lemma

$\partial_1 \partial_2 (ABC) = \partial_1 (AB + BC + AC) = A + B + B + C + C + A = 0$
$= 2A + 2B + 2C.$

$\boxed{\operatorname{Im} \partial_2 \subseteq \ker \partial_1}$

$V, \mathcal{E}, F$ vectors spaces over $\mathbb{Z}_2$ generated by vertices, edges, faces.

Define $H_1(\Sigma, \mathbb{Z}_2) := \dfrac{\ker(\partial_1 : \mathcal{E} \to V)}{\operatorname{Im}(\partial_2 : F \to \mathcal{E})}$

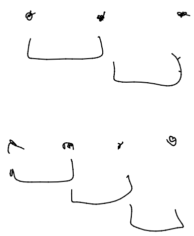
$\dim H_1 = \dim(\ker \partial_1) - \dim \operatorname{Im}(\partial_2)$

rank nullity $(\dim \mathcal{E} - \operatorname{rank} \partial_1) - (\dim F - \dim(\ker \partial_2))$

nullity $\Big($

$$\left(\dim \Sigma - \operatorname{rank} \partial_2\right) - \left(\dim F - \dim(\ker \partial_2)\right)$$

Note: (1) Σ is connected $\Rightarrow$ $\operatorname{Im} \partial_1 : E \to V$ consists of sum of even number of points



$$\Rightarrow \dim V = 1 + \operatorname{rank} \partial_1$$

(2) $\ker(\partial_2 : F \to E)$ consists of sum of all faces

$$\Rightarrow \dim \ker \partial_2 = 1$$

Now: $\dim H_1 = 2 - V + E - F = 2 - \chi(\Sigma)$

What's missing? $\dim H_1$ is a topological invariant that does not depend on subdivision.

faces are D_2 → simply connected.

idea: Any element in $\ker \partial_1 : C_1 \to C_0$, can be replaced by an linear combination of edges of a subdivision with something in ∂C_2.