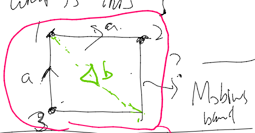
590 W7

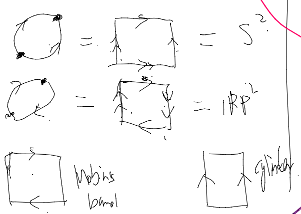
Friday, March 12, 2021 12:01 PM

Cut and paste

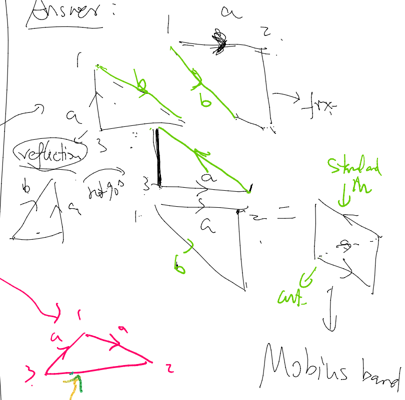
what is this?



Recall: Simple surfaces



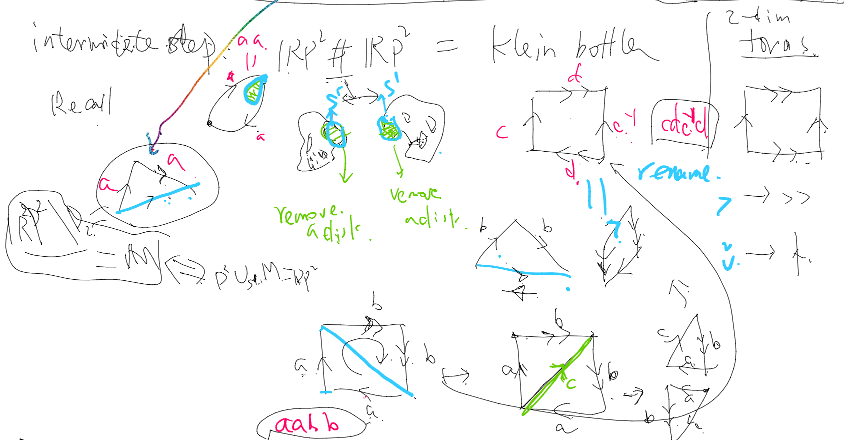
Answer:



goal: find Q in HW4: $T^2 \# \mathbb{RP}^2 =$ ~~$\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2$~~

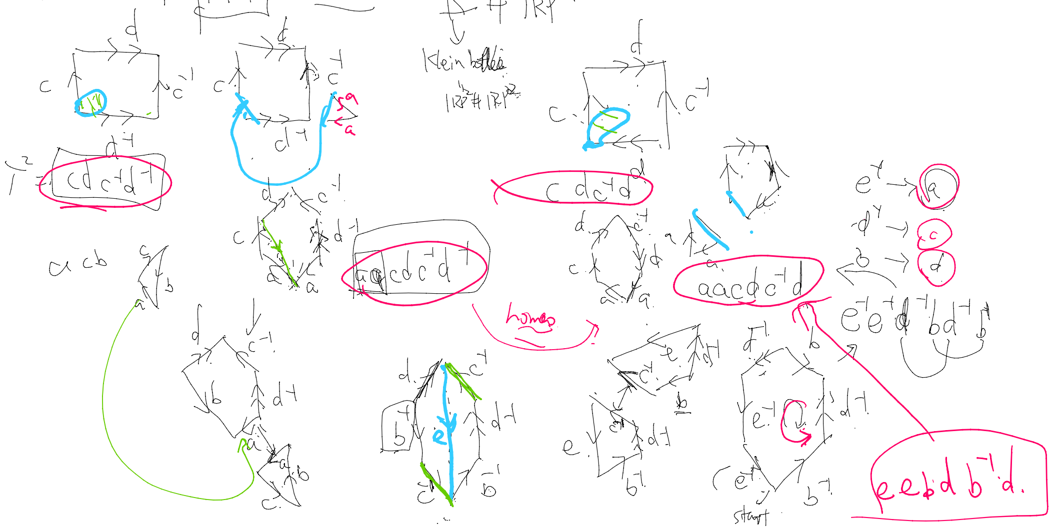
intermediate step: $\mathbb{RP}^2 \# \mathbb{RP}^2$ = Klein bottle

2-dim torus

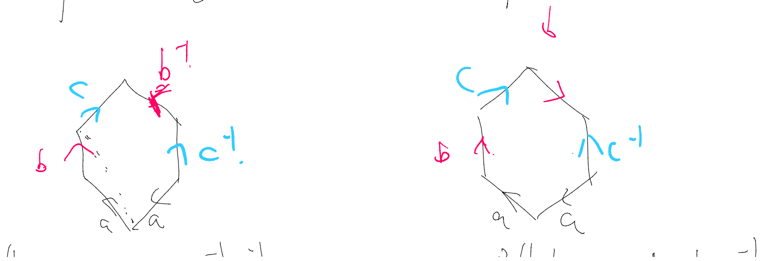


Prove. $T^2 \# \mathbb{RP}^2$ homeo $K \# \mathbb{RP}^2$

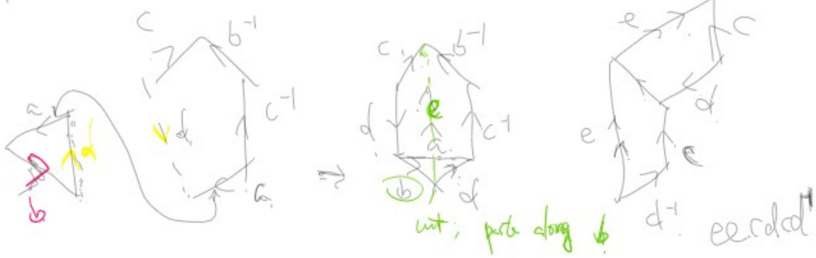
Klein bottle $\mathbb{RP}^2 \# \mathbb{RP}^2$



The following is a more precise picture



$P\#T = aabcb^{-1}c^{-1}$ $\qquad$ $P\#K = aabcbc^{-1}$



Topology: key questions:

Yes ← ① Whether two spaces are homeomorphic?

No ← ② For a class of object (topological spaces), how to classify up to homeo?

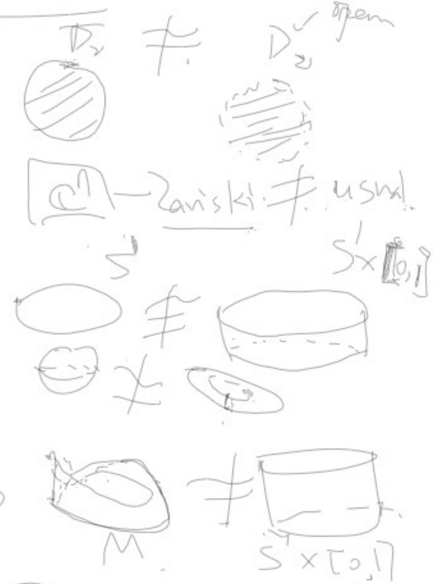
① Yes, easier → explicit construction (homeomorphism)

① No, hard → need to prove no homeo.

↳ invariant properties

- compactness
- separation properties
- connectedness
- Euler number
- orientability

remove two pts from S^1



1860's → 1900's poincaré

② Classification of topological surfaces → 2-dim manifold

S is either S^2, $\# T^2$ (orientable, $\chi = 2-2n$)

$\# T^2 \# \mathbb{RP}^2$ non-orientable

orientability, Euler number are complete invariant for S

No complete (easy)

are complete invariant for S

orientable $\chi = 2 - 2n + 1$

No complete invariant

1900's Poincaré Q: How about M^3 → 3-dim manifolds?

what's a good characterisation of S^3

Poincaré Conjecture. (2003)

Advanced invariants → algebra → group

G → set, $\cdot$ → multiplication → closedness $(a \cdot b) \in G$

s.t. ① associative $(a \cdot b) \cdot c = a (b \cdot c)$

② unit element e, i.e. $e \cdot a = a \cdot e = a \quad \forall a \in G$

③ inverse, $\forall a, \exists a^{-1}$ s.t. $a \cdot a^{-1} = a^{-1} a = e$.

fundamental group & homology groups

correct. today.

Poincaré: connected closed M^3 has the same fundamental group as S^3, then $M^3 \cong S^3$

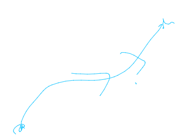
wrong!

connected closed M^3 whose homology is the same as S^3, then M^3 homeo S^3.

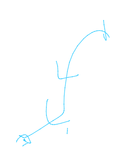
Recall: continuous path $f: I \to (X, x)$, $I = [0,1]$, continuous

map

loop: f — — — s.t. $f(0) = f(1)$



inverse of a path $f^{-1}(t) := f(1-t)$ → $f^{-1}(1) = f(0)$, $f^{-1}(0) = f(1)$



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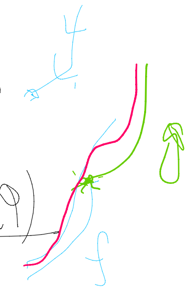
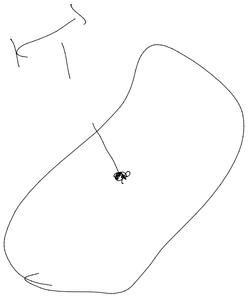
inverse of a path $f^{-1}(t) := f(1-t) \longrightarrow f^{-1}(1) = f(0)$, $f^{-1}(0) = f(1)$



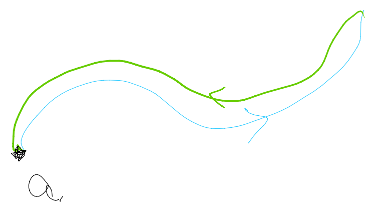
multiplication ← (*) → concatenation $f : [0,1] \to X$, $g : [0,1] \to X$ s.t. $f(1) = g(0)$



$$f * g = \begin{cases} f(2t) & 0 \le t \le \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$

unit element : Def : For a given point $a \in X$, define

$e_a : I \to X$ s.t. $e_a(t) \equiv a$

$f * f^{-1}$  $\neq e_a$

Conclusion : space of path, even with fixed initial point,

dose NOT form a group, w.r.t. * and e_a

We need homotopy relation...

↓

$\boxed{\text{continuous deformation}}$

$f_0 : X \xrightarrow{\text{continuous}} Y$

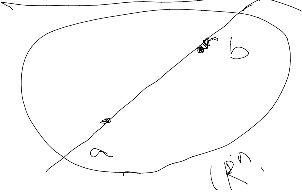
Def : (X, τ_X), (Y, τ_Y) are topological spaces, $f_1 : X \to Y$ continuous

$f_0 \overset{\text{homotopic}}{\sim} f_1$ if there exist a continuous map

$\underset{x \in X,\ t \in [0,1]}{H(x,t)} : X \times I \to Y$ s.t. $H(x,0) = f_0(x)$

$H(x,1) = f_1(x)$.

Straight line homotopy

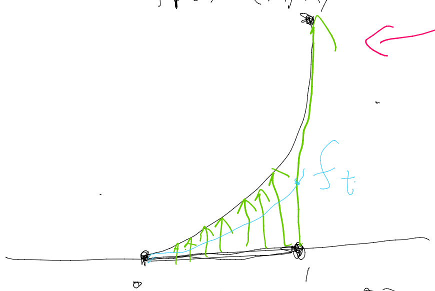


$f(t) = at + (1-t)b$

$t = 1 \to a$

$t = 0 \to b$

$f_1(x) = (x, x^3)$



$f_0(x) = (x, 0)$

$f_0(x)$

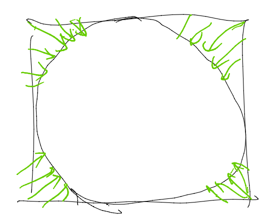
$f_1(x)$

paths in $\mathbb{R}^n$

$F := t f_1(x) + (1-t) f_0(x)$

homotopy b/w f_0 and f_1

homotopic



Def (homotopy equivalence)

weaker equ relation than homeo

(X, T_X), (Y, T_Y) are homotopic equivalent

if there exists functions $f: X \to Y$, $g: Y \to X$
continuous

s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

homotopic   homotopic