LOG(M)

Curves in Surfaces and Mapping Class Group

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Introduction

Goal

To understand constructions of psuedo-Anosov maps and possibly produce a novel construction.

Pseudo-Anosov maps are a particular type of homeomorphisms of topological spaces known as *surfaces*. Homeomorphisms are of utmost interest to topologists, and pseudo-Anosov maps are some of the more complicated ones.

Definition. *A surface is a topological space that behaves locally like* $\mathbb{R}^2$.

The following are common examples.

Examples:

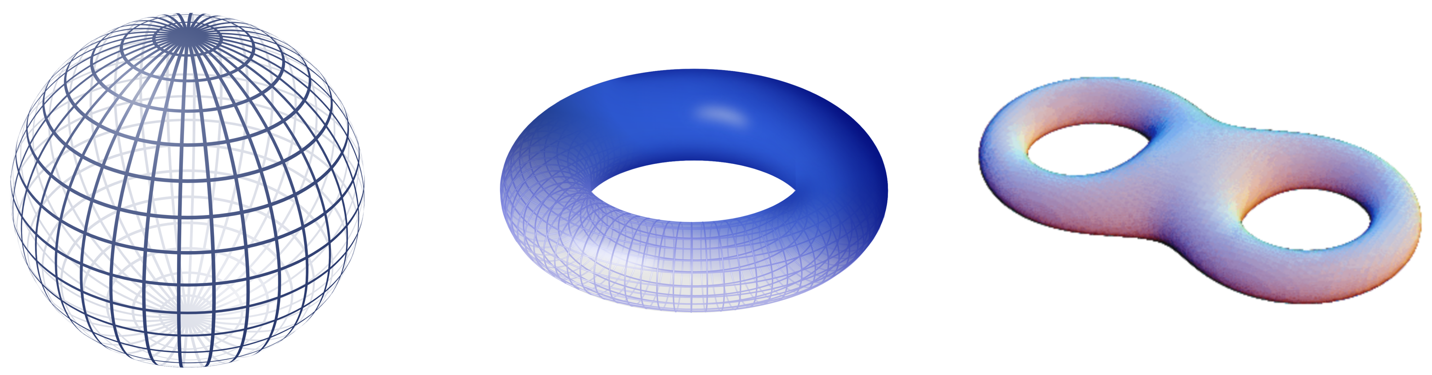


Figure 1: *A sphere [3], a torus [5], and a double torus [9].*

Preliminary and Background

Definition 1: ($\mathrm{Homeo}^+(S)$, Mapping Class, Mapping Class Group)

Given a compact, orientable surface S,

- $\mathrm{Homeo}^+(S)$ is the set of orientation-preserving homeomorphisms from S to itself.
- A *mapping class* of S is a homotopy class of homeomorphisms from S to itself.
- The *mapping class group* of S is the group of mapping classes of S.

Theorem 1: (Nielson-Thurston Classification of Mapping Classes)

Given a compact, orientable surface S, let h be a homeomorphism from S to itself. The at least one of the following is true:

- h is *periodic*, i.e. some power of h is the identity.
- h is *reducible*, i.e. h preserves some finite union of disjoint simple closed curves on S.
- h is *pseudo-Anosov*, i.e. no power of h fixes any curve on S.

Theorem 2: (Dehn)

All mapping classes can be written as products of Dehn twists.

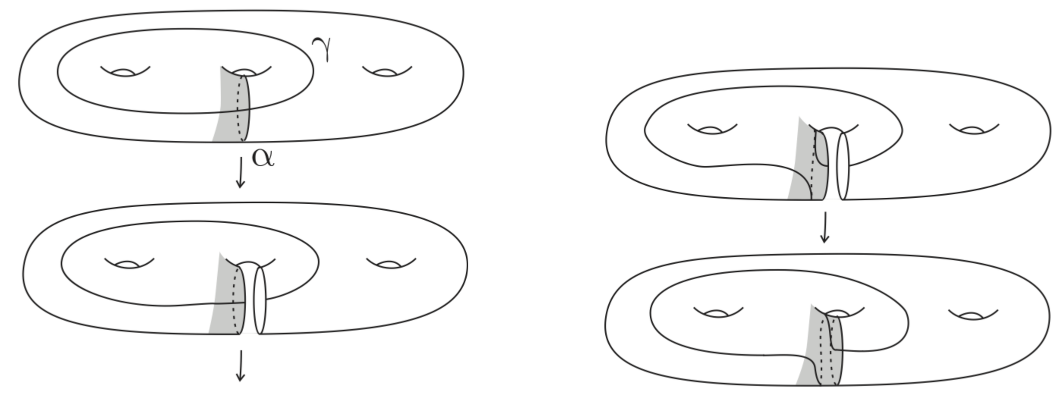


Figure 2: *Demonstration of Dehn Twist*

Theorem 3: (Penner's Construction)

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be multicurves on a surface S in minimal position such that $(A \cup B)^c$ is a union of disks and once punctured disks. Then any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov, provided that all $n + m$ Dehn twists appear in the product at least once.

This construction is fairly general, so Penner made the following conjecture.

Conjecture (Penner, 1988)**.** *Every pseudo-Anosov mapping class has a power that arises from Penner's construction.*

This has been disproved due to Shin-Strenner.

Methods and Results

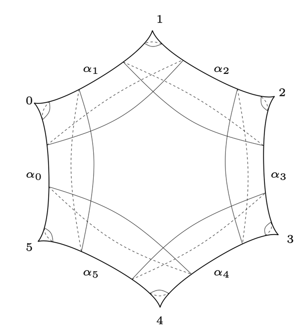
Construction of a pseudo-Anosov Map

Verberne (2019) presents an method to construct pseudo-Anosov maps, some of which not coming from Penner's Construction.

Theorem (Verberne, 2019). *On $S_{0,6}$, the composition of Dehn twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$*

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2$$

is a pseudo-Anosov map, where D_i is a half Dehn twist around α_i.



A pseudo-Anosov Map Visualized in Polygonal Representation

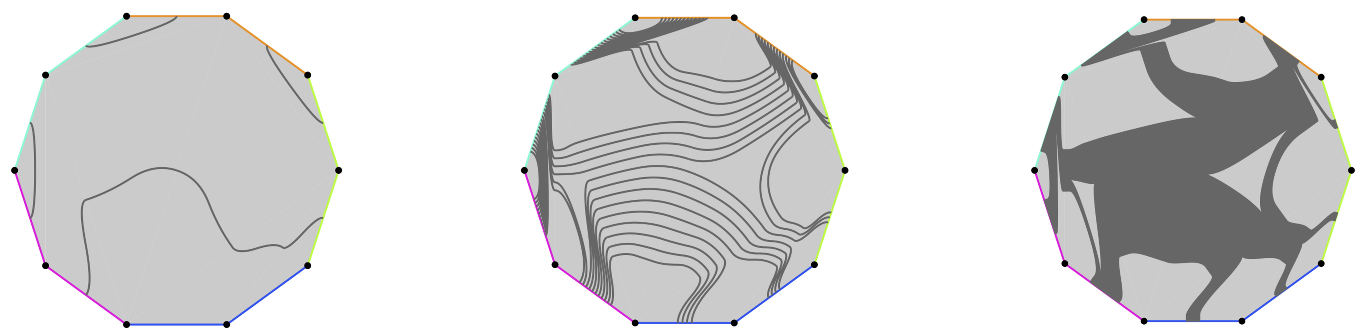


Figure 3: *A curve, its image under ϕ, and that under ϕ^2 on an six-times-punctured sphere.[2]*

In fact, no curve on the surface is preserved under ϕ^n for all $n \in \mathbb{N}$. This is a characterization of pseudo-Anosov maps.

Pseudo-Anosov Maps Coming from Half Twists

Conjecture. *There is an analogue of Penner's construction for half twists.*

Example. *On $S_{0,6}$, the composition of half Dehn twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$*

$$\phi = D_0 D_3^{-1} D_1 D_4^{-1} D_2 D_5^{-1}$$

is a pseudo-Anosov map.

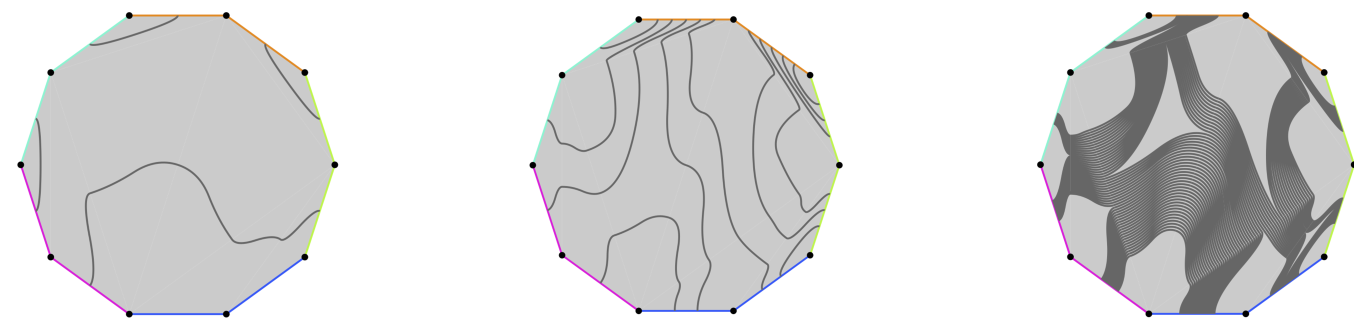


Figure 4: *A curve, its image under ϕ^4, and that under ϕ^8 on an six-times-punctured sphere.[2]*

However, the direct translation of the full twist version does not hold. Hence we ask Question 1 and will further probe in this direction.

Counterexample. *On $S_{0,6}$, the composition of half twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$*

$$\phi = D_0 D_3 D_1 D_4^{-1} D_2 D_5^{-1}$$

is reducible.

Future Directions

These constructions using curves on surfaces contain gaps, i.e. they do not give every pseudo-Anosov map for an arbitrary surface. In particular, there are two questions our team would like answered.

Question 1 In general, how to construct pseudo-Anosov maps using half twists on punctured spheres?

Question 2 Does there exist a construction using train track methods such that every pseudo-Anosov map comes from it?

Answering Question 1. We are looking for a more precise analogue of Penner's construction using half twists.

Answering Question 2. This will not easily be answered. It is already known that Penner's construction does not give all pseudo-Anosov maps, but maybe one or more of the current constructions give all of them.

References

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