Curves in Surfaces and the Mapping Class Group

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Introduction and Definition

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- Surface
  Intuition: objects such that they resemble $\mathbb{R}^2$ when "zoomed in."

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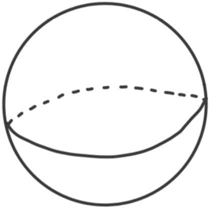

(a) Sphere

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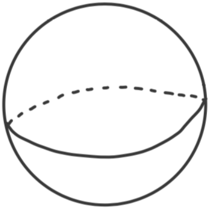

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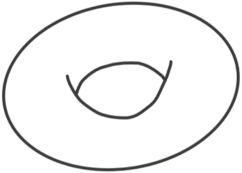

(b) Torus

Introduction and Definition

- Symmetry

  Intuition: repeated sameness.

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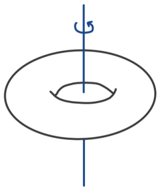

Figure: Example of symmetry: rotate π counterclockwise

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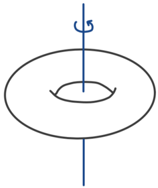

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- Homeomorphism
  Continuous bijection with continuous inverse.

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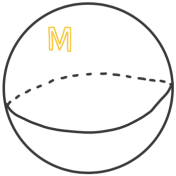

(a) Example of curve on sphere

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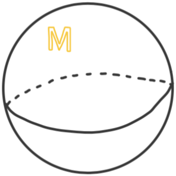

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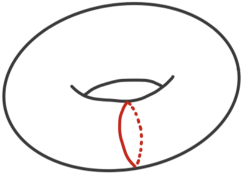

(b) Example of curve on torus

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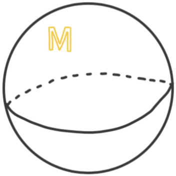

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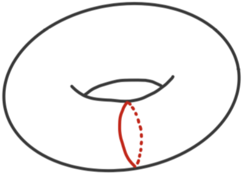

(b) Example of curve on torus

Note: We study *closed* curves, which are images of circles under homeomorphisms.

History and Motivation

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- Dehn [1920s]

$$\mathrm{Homeo}^{+}(S) \rightsquigarrow \mathrm{Mod}(S)$$

  - $\mathrm{Homeo}^{+}(S)$: The set of orientation-preserving homeomorphisms that map S to S.

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  - "Sameness": Two homeomorphisms are the "same" if for every curve in the domain, the image of the curve under one homeomorphism can be "wiggled" into the image of another.
  - $Mod(S)$: The "mapping class group" of S.

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- Nielson & Thurston
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  Classified $\mathbb{S}$ into three types of elements:
  1. Finite-order elements
  2. Reducible elements
  3. Pseudo-Anosov maps

Classification

- Finite-order elements
  Some power fixes *all* curves.

Classification

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Figure: Example of a finite order element: rotate clockwise by 2π

Classification

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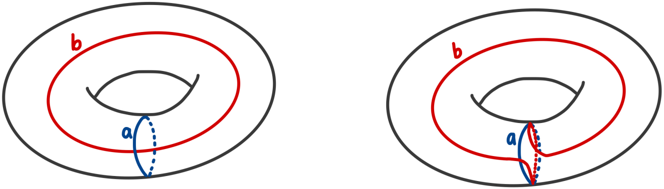

Figure: Example of reducible element: Dehn twist

Classification

- Pseudo-Anosov maps
  No power fixes *any* curve.

Classification

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- Pseudo-Anosov maps
  No power fixes *any* curve.

Figure: Too complicated to display

Motivation and Project Goals

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- Standard construction for pseudo-Anosov maps from S to S – *Penner's Construction*

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- Standard construction for pseudo-Anosov maps from S to S – *Penner's Construction*
- There exist constructions of pseudo-Anosov maps that do not come from Penner's Construction or its powers.
- Our goal: visualizing and understanding these new constructions. Trying to create new constructions of pseudo-Anosov maps.

A Torus

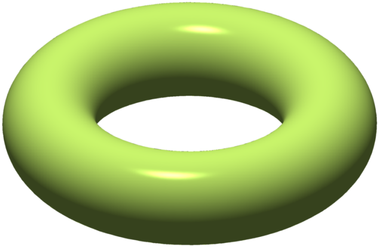

Figure: A Torus (Retrieved from https://en.wikipedia.org)

A Connected Sum of Two Tori

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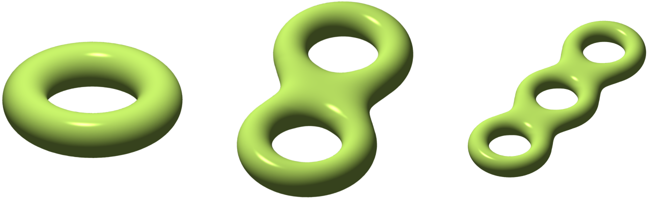
Classification of Surfaces

Theorem

Every orientable compact surface is homeomorphic to a sphere or a connected sum of tori.



Dehn Twist

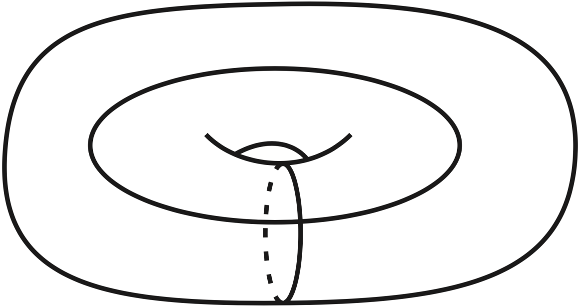


Figure: (Retrieved from *Office Hours with a Geometric Group Theorist* by Matt Clay and Dan Margalit)

Polygonal Representation of a Torus

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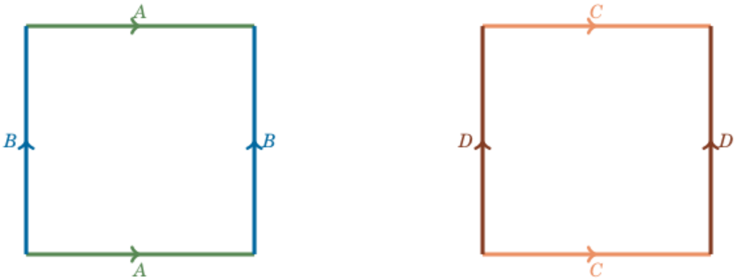

Figure: Polygonal Representation of a Torus (Retrieved from https://math.stackexchange.com)

Polygonal Representation of a Connected Sum of Two Tori

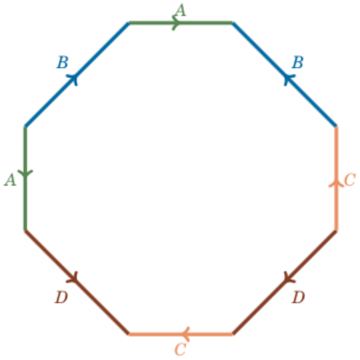

Figure: A Connected Sum of Two Tori (Retrieved from https://math.stackexchange.com)

Dehn Twist in Polygonal Representation

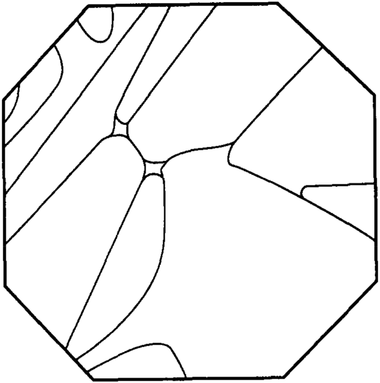

Figure: A pseudo-Anosov map that is a Composition of Dehn Twists (by M. Bestvina and M.Handel)

Linear Algebra and Mod(S)

- Elements of Mod(S) have matrix representations
- e.g. Mod(T^2) $\cong SL_2(\mathbb{Z})$.

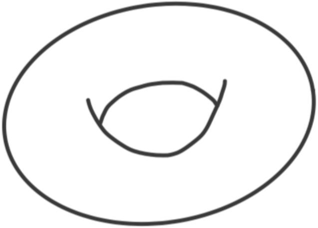

Figure: T^2=Torus

Linear Algebra and Mod(S)

- Convert curves to a graph
- Can think of curves a and b as a "basis"
- Get matrix representations of Dehn twists about a and b, respectively:
  - $[T_a] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $[T_b] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

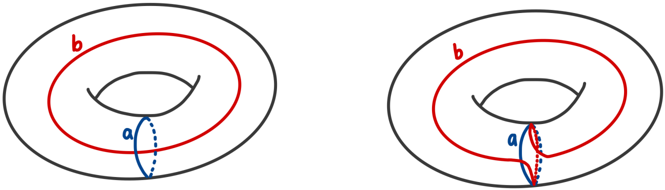

Figure: Dehn twist about a

Linear Algebra and Mod(S)

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If we consider, $A = [T_a][T_b] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, then A has eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2} = 1/\lambda_1$, with corresponding eigenvectors $v_1 = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{bmatrix}$, respectively.

Linear Algebra and Mod(S)

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- So A stretches a direction v_1 by the factor λ_1, and it squeezes another direction v_2 by $\lambda_2 = 1/\lambda_1$.
- This is an example of a pseudo-Anosov map!
- Every pseudo-Anosov exhibits this property

Which One?

- Elements of Mod(S) are called classes and denoted $[f]$, where $[f] := \{\text{homeos that are the "same" as f}\}$.
- Q: Given a class $[f] \in \text{Mod}(S)$, what is the "best" element in the class to analyze?
- A: The one whose matrix representation has the lowest eigenvalues!

Bestvina-Handel

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The Bestvina-Handel algorithm will find the “best” representation of a class for us!

- Lowers eigenvalues to find the best map
- Does so by changing the graph of the image of a curve
- Examples: Valence 2 isotopy, folding, and subdivision

What's Next?

Tasks/questions yet to be completed/answered

- Use the Bestvina-Handel algorithm to study the maps given by Penner's construction
- How do new constructions of pseudo-Anosov maps differ from Penner's, and why do they give different pseudo-Anasov maps?
- Compute the stretch factor of a pseudo-Anosov map from a different construction. Does it differ from that of Penner.
- Can we create a new construction?

References

[1] Bestvina, Mladen, and Michael Handel. *Train-tracks for surface homeomorphisms*. Topology 34.1 (1995): 109-140.

[2] Clay, Matt, and Dan Margalit, editors. *Office Hours with a Geometric Group Theorist*. Princeton University Press, 2017.

[3] Hitchin, Nigel. *Geometry of Surfaces*. Course notes. 2013.

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