ISOTOPY OF SYMPLECTIC SPHERES AND $\pi_1 Ham(M,\omega)$ OF RATIONAL SURFACE

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ABSTRACT. A positive rational surface is diffeomorphic to n-point blow up of $\mathbb{C}P^2$ with a symplectic form ω such that ω is c_1-positive. We determine the rank of $\pi_1(Ham)$ for a symplectic form ω of type $\mathbb{A}$ and $\mathbb{D}$ on a positive rational surface. As an application, we obtain isotopy results for symplectic spheres and stability results for the fundamental groups of space of ball packing in $\mathbb{C}P^2$.

CONTENTS

1. Introduction

Let (M, ω) be a symplectic manifold. The topology of the diffeomorphism group and its subgroups, such as the symplectomorphism group, has long been a fascinating subject in mathematics. Of particular interest is the identity component of the symplectomorphism group, which is equivalent to the Hamiltonian diffeomorphism group for simply connected symplectic manifolds.

The study of the fundamental group of the Hamiltonian diffeomorphism group ($\pi\_1(\mathrm{Ham}(M, \omega))$) has attracted much attention due to the Seidel representation (cf. [Sei97]) from the group to the quantum homology ring, as well as the existence of the bi-invariant Hofer metric (cf. [Pol98]) on the group. In particular, the rank of the fundamental group plays an essential role in understanding the Hofer diameter conjecture and non-displaceable Lagrangian tori.

We will focus on positive rational surfaces. Recall that a symplectic rational surface (X, ω) is *positive* if $c\_1(X) \cdot [\omega] > 0$. The positivity condition of a rational surface is equivalent to the existence of a divisor $D \subset X$, such that (X, D) is a log Calabi-Yau surface.

Our recent works [LL20; LLW22] have made progress in understanding the topology of the symplectomorphism group for rational symplectic manifolds with small Euler numbers. In particular, we utilized the generalized Alexander duality in [Eel61] to compute the rank of the fundamental group of the symplectomorphism group.

Based on our previous results, we put forward the following conjecture:

Conjecture 1.1. *For a rational symplectic manifold (X, ω) where the root system associated with its Lagrangian spherical classes is of type $\mathbb{A}$ or $\mathbb{D}$, we have:*

$$(\chi(X)-2)(\chi(X)-3)/2 = r^+[\pi_0(Symp(X,\omega))]+\mathrm{rank}[\pi_1(Symp(X,\omega))]-\mathrm{rank}[\pi_0(Symp_h(X,\omega))]. \tag{1}$$

Here, $r^+[\pi_0(Symp(X, \omega))]$ denotes the number of positive roots for π_0 as a Weyl group. It is worth noting that a symplectic class can be endowed with a *type* using the root system associated with its Lagrangian spherical classes.

In [LLWxi], we made significant process on $\pi_0(Symp_h(X, \omega))$ of a positive rational surface. It is trivial if it is of type $\mathbb{A}$ and is a sphere braid group if it is of type $\mathbb{D}$. Building on this result and using the filling divisor technique in [LLWxi], we prove the following theorem:

Theorem 1.2. *Conjecture 1.1 holds for positive rational surfaces X with type $\mathbb{A}$ or $\mathbb{D}$ symplectic forms. Moreover, for a reduced form, it is completely determined by a finite collection of -2 symplectic sphere classes of degree 0, 1, and 2. Here degree of a class $A \in H_2(X, \mathbb{Z})$ means $A \cdot H$, where H is the hyperplane class in $\mathbb{C}P^2$.*

Our result provides a deeper understanding of the topology of the symplectomorphism group for positive rational surfaces and has implications for various problems. Our result confirms that the upper bound given by McDuff in [McD08] for $\pi_1(Symp(X, \omega))$ often gives the precise rank, but in the case of type $\mathbb{D}$ forms, the actual rank is much smaller than the upper-bound. We also have the following corollary, on the isotopy uniqueness of symplectic sphere classes. Notice that previously, such uniqueness is only known for exceptional spheres or spheres with positive self-intersections. Those proofs are the genericity arguments in Gromov-Witten theory, while ours has a completely different nature.

Corollary 1.3. *For a positive rational surface,*

- *any homologous symplectic -2 spheres are symplectically isotopic.*
- *when the form is of type $\mathbb{A}$, any symplectic spheres which are proper transforms of the conic (in class $2H - E_1 - \cdots - E_n$), are symplectically isotopic.*

Note that homological actions are generated by symplectic (-2) spheres. For a type $\mathbb{A}$ or $\mathbb{D}$ symplectic form,[LLWxi] showed that the SMCG is trivial or braid group, which is torsion-free. In alignment with this result and [CLW21], we have the following corollary:

Corollary 1.4. *For a positive rational surface with a type $\mathbb{A}$ or $\mathbb{D}$ symplectic form, the finite group acting symplectically is just the subgroup of homological actions.*

Notice that the main theorem 1.2 hints that the stability result of [Anj+ub] can be extended to the c_1-positive cone for an arbitrary rational surface. We conjecture that even if there are infinitely many negative sphere classes, only a finite subset of them will contribute to the change of homotopy type of $Symp(X,\omega)$:

Conjecture 1.5. *For a fixed number $k \in \mathbb{Z}^+$, there are finitely many walls given by sphere classes with negative self-intersections, such that: when ω_1 and ω_2 belongs to the same chamber of the symplectic cone of M, $\pi_i(Symp(M,\omega_1)) = \pi_i(Symp(M,\omega_2)), \forall 1 \leqslant i \leqslant k$.*

For rational surfaces with Euler number at most 12, [Anj+ub] proved this conjecture in full generality. It is much more difficult to approach when we blow up $\mathbb{C}P^2$ at more than 9 points, due to the abundance of curves and Nagata conjecture. For positive rational surfaces, [LLWxi] proved the stability for $\pi_0(Symp)$, where the wall is characterized by Lagrangian spherical classes. In this paper, we prove this for $\pi_1(Symp)$, where the wall is the same as π_0. We give a new characterization by $(H-E_1)$-fiber classes and $(H-E_1)$-section classes, as detailed in Section 4.

Meanwhile, the partial solution of Conjecture 1.5 implies the stability for the fundamental group of space of unparametrized ball packings $\mathrm{Emb}^*_\omega(B^4(\vec{c}), M)$, where M^4 is rational:

Proposition 1.6. *If the blowup form $\tilde{\omega}$ is c_1 positive, then the π_1 of $Emb^*_\omega(B^4(\vec{c}), M)$ is invariant if the change of size $\vec{c}$ does not affect Lagrangian sphere classes of the blowup symplectic form.*

It is also interesting to consider the question of whether one can represent the free loops in $\pi_1(Symp(M,\omega))$ by circle actions. Notice that when $M = X_5$, [Anj+23] observed that there are loops that are not represented by Hamiltonian circle actions. Moreover, under deformation, some loops generated by circle actions become loops that are not circle representatable. We give such examples for any X_k in Example 6.4. Further, inspired by this example and the spirit of the above stability results and conjecture, we end up with the following conjecture:

Conjecture 1.7 (=Conjecture 6.7). (1) *X_k with a symplectic form in class $[1, a, \frac{1-a}{2}, \frac{1-a}{2}, \cdots, \frac{1-a}{2}]$ has Hamiltonian circle actions if and only if $a > \frac{k-3}{k-1}$.*

(2) *Moreover, for X_k with a reduced symplectic form, that is c_1-nonpositive, or is c_1-small ($c_1 \cdot [\omega] = \epsilon$), any Hamiltonian loop cannot be represented by Hamiltonian circle actions.*

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2. Symplectic cone and special holomorphic curves

For a rational surface, Mcduff [McD98] asserts that the orientation-preserving diffeomorphism group $\mathrm{Diff}^+(X)$ acts on the symplectic cone $\mathcal{C}_X$. As diffeomorphic symplectic forms have homeomorphic symplectomorphism groups, we can restrict our attention to a fundamental region of $\mathcal{C}_X$ under the action of $\mathrm{Diff}^+(X) \times \mathbb{R}^+$.

In this section, we recall the explicit description of $V(X)$ and its c_1-positive portion $N\mathbb{R}$, together with the root system of a reduced symplectic form. Now we will assume that the rational surface $X_n = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ is framed, i.e. has a choice of basis $H, E_1, \cdots E_n$ of $H_2(X_n, \mathbb{Z})$.

2.1. **The c_1-positive symplectic cone.** Let $K \in H^2(X;\mathbb{Z})$ be the symplectic canonical class of some orientation-compatible symplectic form on X. Recall the K-symplectic cone

$$\mathcal{C}_{X,K} = \{e \in \mathcal{C}_X \mid e = [\omega] \text{ with } K_\omega = K\}. \tag{2}$$

We also highlight the following c_1-positive part:

$$P_K := \{e \in \mathcal{C}_K | e \cdot (-K) > 0\} \tag{3}$$

It is called c_1-nef cone in [LZ15].

First, we can use the $\mathbb{R}^+$ action to normalize the area $e(H) = 1$. A class in $\mathcal{C}_{X,K}$ is represented by the vector $(1, |m_1, \cdots, m_n)$ or simply by $(m_1, \cdots, m_n)$.

Recall that such a vector $(\nu | a_1, \cdots, a_n)$ is reduced if $\nu \leq a_1 + a_2 + a_3, and a_1 \geq \cdots \geq a_n > 0$. We define the **normalized reduced symplectic cone** $V_n = V(X_n)$ for $X = X_n$ as the subset of reduced symplectic classes with $\nu = 1$ in $\mathcal{C}_{X_n}$. Recall from [LL02; KK17] that any class $e \in \mathcal{C}_{X_n}$ has a unique reduced representative in its $\mathrm{Diff}^+(X_n) \times \mathbb{R}^+$ orbit.

We use the shorthand notation

$$(1| \underbrace{a_1, \cdots a_1}_{l_1}, \cdots, \underbrace{a_k, \cdots a_k}_{l_k}, \cdots, \underbrace{a_m, \cdots a_m}_{l_m}) := (1| \ a_1^{\times l_1}, \cdots, a_k^{\times l_k}, \cdots, a_m^{\times l_m}).$$

When $l_i = 1$, we will omit it in the notation.

To describe $V(X_n)$ explicitly, we consider the closed polytope $\overline{\mathcal{R}}_n$ in $H_2(X_n;\mathbb{R})$ with the following $n+1$ vertices:

$M = -\frac{1}{3}K,\ \ O = (1| \ 0^{\times n}),\ \ A = (1| \ 1, 0^{\times n-1}),\ \ G_3 = (1|\frac{1}{2}, \frac{1}{2}, 0^{\times n-2}),$

$G_4 = (1|\ (\frac{1}{3})^{\times 3}, 0^{\times n-3}), \cdots, G_n = (1|\ (\frac{1}{3})^{\times n-1}, 0)$.

Lemma 2.1. *For $X = X_n$, a class σ is normalized and reduced if and only if $\sigma \in \overline{\mathcal{R}}_n$. When $n \leq 9$, the convex hall of $M, G_1, \cdots, G_n$ is the normalized reduced cone. Moreover, when $n \geq 10$, the normalized reduced symplectic cone is the intersection of the convex hall $M, G_1, \cdots, G_n$ intersected with the positive condition $\sigma^2 > 0$. V_{n+1} can be defined inductively by adding n new vertices from V_n.*

For $n \geq 10$, the general formula for the n new vertices is as follows:

On the $G_{n+1}O$ edge with $O = (1|\ 0^{\times n})$: $(1|\ (\frac{3}{n})^{\times n})$.

On the $G_{n+1}A$ edge with $A = (1|\ 1, 0^{\times n-1})$: $(1|\frac{n-7}{n-3}, (\frac{2}{n-3})^{\times n-1})$.

On the $G_{n+1}G_3$ edge with $G_3 = (1|\ \frac{1}{2}, \frac{1}{2}, 0^{\times n-2})$: $(1|\ \frac{n-5}{2(n-3)}, \frac{n-5}{2(n-3)}, (\frac{2}{n-3})^{\times n-2})$.

On the $G_{n+1}G_i$ edge for $4 \leq i \leq 10$: $(1|\ (\frac{1}{3})^{\times i}, (\frac{9-i}{3(n-i)})^{\times n-i})$.

For X_n, recall that the -2 sphere classes form a **topological root system** Γ_K, where the simple roots of Γ_K are given by l_i, $0 \leq i \leq n-1$, $l_0 = H - E_1 - E_2 - E_3$, $l_i = E_i - E_{i+1}, \forall i > 0$.

Lemma 2.2. *Suppose ω is a normalized reduced symplectic form on X_n, then the Lagrangian spheres classes of ω form a class of type $\mathbb{A}$ $\mathbb{D}$ or $\mathbb{E}$:*

(1) *ω is of type $\mathbb{E}_k, k = 6, 7, 8$ (or simply type $\mathbb{E}$) if $[\omega] = (1|\ (\frac{1}{3})^{\times k}, m_{k+1} \cdots)$, where $m_{k+1} < \frac{1}{3}$, $k = 6, 7, 8$ respectively. The subspace of type $\mathbb{E}_k$ classes is of codimension k in the reduced cone.*

(2) *ω is of type $\mathbb{D}_k, k \geq 4$ (or simply type $\mathbb{D}$), if*
$$[\omega] = (1|\ a, (\frac{1-a}{2})^{\times k}, m_{k+2} \cdots), \ \textit{where}\ \frac{1-a}{2} > m_{k+2}\ \&\ \frac{1}{3} < a < 1;$$
or
$$[\omega] = (1|\ (\frac{1}{3})^{\times k}, m_{k+1} \cdots), \ \textit{and}\ k = 5.$$
The subspace of type $\mathbb{D}_k$ classes is of codimension k in the reduced cone.

(3) *ω is of type $\mathbb{A}$ for all other possibilities. Equivalently, ω is of type $\mathbb{A}$ if and only if at least one of l_i has positive ω-area for $i = 0, 2, 3, 4$ (See Figure 1).*

2.2. **A positive sphere cone Lemma relative to (-2) sphere classes.** The aim for this part is Lemma 2.4. It is a relative version of positive sphere cone Lemma parallel to Lemma 5.24 in [LZ15].

Lemma 2.3. *For any $\omega \in P_{K^+}$, we can find a positive combination of positive sphere classes in S_K^+ by Lemma 2.3.*

We will show that this holds for $P_{K^+}^D$ and S_K^{D+}, which is the positive spheres pair D positively. This relative version is useful in the stability result for $\pi_1(Symp)$, and it plays the same role as the original version in the stability for $\pi_0(Symp)$ in [LLWxi].

Lemma 2.4. *Let D be a symplectic -2 sphere class. Any class $d \in P^D_{K+}$ can be written as a finite $\mathbb{R}_+$-combination of positive sphere classes in S^{D+}_K, where P^D_{K+} is the relative sympelctic cone making a class D symplectically positive, and S^{D+}_K is the set of positive classes pairing with D non-negative.*

Proof. Notice that P^D_{K+} is the orbit space of P_{K+} under the reflection along the class D. From the convex geometry point of view, this is just to say that the hyperplane defined by $D \cdot - = 0$ divides P_{K+} into symmetric half cones.

For $3 \le k \le 8$, We do induction. Suppose we have done the case for $3 \le k \le l-1$, we want to argue that for $M_l = \mathbb{CP}^2 \# l\overline{\mathbb{CP}^2}$, $\mathcal{S}^{D+}_K = P^D_{K+}$.

Recall Lemma 5.24 (1) in [LZ15], P_K is an open polytope with each face of the boundary a wall of a class in $\mathcal{E}_{M_l,K}$. Notice that when $k \ge 3$, all the exceptional classes are equivalent up to Cremona transforms.

Also, Lemma 5.24 (1) in [LZ15] can be upgraded to be compatible with the reflection along D: when looking at P_K at the positive side (negative side respectively) of the hypersurface $D \cdot \omega = 0$ for $\omega \in P_K$, the faces of the boundary is a wall of class in $\mathcal{E}_{M_l,K}$ that pair D positive (negative respectively).

Hence e could be written as a finite combination $\sum_{i=1}^q a_i e_i$ with e_i in a boundary face and $a_i > 0$. Notice each boundary face $F_{E'_i}$ (with $E'_i \in \mathcal{E}_{M_l,K}$) of $P_{M_l,K}$ corresponds to $P_{M_{l-1},K}$ by Lemma 5.24 (2) in [LZ15]. Also, when e is on the positive side of the hypersurface defined by D, by convexity, all e_i can be chosen from the positive side of boundary faces intersecting with D.

Then by induction assumption, each $e_i \in \mathcal{S}^{D+}_{M_{l-1},K}$. Hence $e \in \mathcal{S}^{D+}_K$ as well by definition.

When $k \ge 9$, we still do induction. We need to deal with the wall contributed by the class $-K$ in P^D_{K+} and P_{K+}.

Let us start from P_{K+}. Consider a class of the form $V_a = aH - \sum_{i=1}^l E_i$. Given any $e \in P_K$, we can find $a < 3$ such that $e \cdot (aH - \sum E_i) = 0$. This is because $V_3 = -K$ pairs positively with e, and when $a = 0$, V_0 pairs negatively with e. Notice that $V_a \cdot V_3 < V_3 \cdot V_3 \le 0$, so the hypersurface of V_a does not intersect the wall of $-K$.

Now if we let e lie on the positive side of the D-hypersurface, i.e. $e \in P^D_{K+}$. Then notice that e has an open neighborhood in $V_a \cap P^D_{K+}$. We can choose a generic line L in this hypersurface such that $e \in L$ and L does not intersect the D-hypersurface. Notice that L will intersect the polytope P_K inside the interior of the boundary faces F_1, F_2 at e_1 and e_2. We further notice that by the choice of L, both e_1 and e_2 pairs positively with D.

Hence we have $e = a_1 e_1 + a_2 e_2$ with $a_i > 0$, and each class $e_i \in P_{M_{l-1},D,K}$. Because each of them lies in the interior of the face F_i, and it pairs D positively. By induction assumption, $e_i \in \mathcal{S}^{D+}_K$. This finishes the proof. □

Remark 2.5. *Using Lemma 2.14 in [LLWxi], one can easily show that d is a positive combined using $\mathcal{E}^{D+}_K$, which is the set of exceptional classes pairing with D non-negative.*

2.3. **Well-founded relation on the c_1-positive cones.** Recall that a binary relation $\prec$ on a set S is **well-founded** if there are no infinite descending chains $\cdots \prec a_i \prec a_1 \prec a_0$, or in other words every descending chain has a minimal element. When $a \prec b$, we call a is a *predecessor* of b. A well-founded relation allows us to do induction in section 5.3.

Now we describe such a relation on the union of c_1-positive cones $\cup_n N\mathbb{R}(X_n)$ and recall how this relation preserves the type of the symplectic form and the symplectic mapping class group.

Definition 2.6. *Let $u_m = (1|\ a_1^{\times l_1}, \cdots, a_k^{\times l_k}, \cdots, a_m^{\times l_m})$ be a symplectic class in the reduced c_1-positive cone $N\mathbb{R}(X_n)$. Recall that this notion implies the sharp inequalities $a_1 > \cdots a_k > \cdots > a_m$. Now we define a relation $\prec$ on $\cup_{n\geq 0} N\mathbb{R}(X_n)$:*

- *If $m > 1$ and $2a_m + a_1 < 1$, then we say $u_{m-1} = (1|\ a_1^{\times l_1}, \cdots, a_k^{\times l_k}, \cdots, a_{m-1}^{\times l_{m-1}}) \subset N\mathbb{R}(X_{n-m_k})$ a predecessor of u_m. Denote $u_{m-1} \prec u_m$.*
- *if $m = 1$ and $a_1 < \frac{1}{3}$, then we say $u_0 = (1)$ on $X_0 = \mathbb{C}P^2$ the predecessor of u_1.*

Lemma 2.7. *The relation $\prec$ in Definition 2.6 is well founded. It preserves the type of the symplectic form and the symplectic Torelli group. More precisely:*

(1) *Each descending chain of $\prec$ has a minimal element.*

(2) *The relation $\prec$ preserves the type of the symplectic form class.*

(3) *The minimal element is either a type $\mathbb{A}$ symplectic class on $X_n, 0 \leq n \leq 4$, a type D_{n-1} class on X_n, a type D_5 class on X_5, or a type E_k class on $X_k, k = 6, 7, 8$.*

(4) *The relation $\prec$ preserves the symplectic Torelli group.*

Proof. (1) Note the predecessor is a blowdown symplectic form class, and any u has only finitely many predecessors. Hence there is a minimal element in this finite set w.r.t to $\prec$.

(2) This follows from Theorem 3.20 and Proposition 3.25(ii) in [LLWxi].

(3) To see this, we analyze where the element u_m does not have a predecessor. When $m = 1$, the only possibilities are $(1, |a_1^{\times} k), a_1 \geq \frac{1}{3}, k \leq 2$, or $(1, |(\frac{1}{3})^{\times} k), 3 \leq k \leq 8$. Note that those are type $\mathbb{A}$ forms on $X_i, 1 \leq i \leq 4$, $\mathbb{D}_5$ on X_5, and $\mathbb{E}_j$ on $X_j, j = 6, 7, 8$. When $m > 1$, the only case is when $2a_m + a_1 = 1$. Note that u_m is reduced and $a_2 \geq a_m$ and $2a_2 + a_1 \leq 1$. Hence the only possibility is $a_m = a_2$, which means $m = 2$ and the form is of type D_{n-1} on X_n. In all other cases, $m = 0$ and the minimal element is the Fubini-Study form on $\mathbb{C}P^2$, which is of type $\mathbb{A}$.

(4) This follows from (2) and Theorem 1.5 of [LLWxi].

□

2.4. **Upper bound of $\pi_1(Symp)$ for type $\mathbb{A}, \mathbb{D}$ forms on $\mathbb{C}\mathbf{P}^2 \# \mathbf{n} \overline{\mathbb{C}\mathbf{P}^2}, \mathbf{n} > 5$.** We will use the blowing down smallest exceptional sphere process.

Note that this process will not change the type of the symplectic form.

Then recall the following from [LLW22]:

Proposition 2.8. *Let (X,ω) be a symplectic rational surface with a given reduced form, $(\widetilde{X}_k,\widetilde{\omega})$ be the blow-up of X at k points, and denote $r=b_2(X)$. Assume that the k blowup is smaller than an arbitrary exceptional class of X.*

If the k blow-up sizes are distinct, then

$$rank[\pi_1(Symp_h(\widetilde{X}_k,\widetilde{\omega})] \le rank[\pi_1(Symp_h(X,\omega))] + rk + k(k-1)/2;$$

and if the k blowup sizes are the same, then

$$rank[\pi_1(Symp_h(\widetilde{X}_k,\widetilde{\omega})] \le rank[\pi_1(Symp_h(X,\omega))] + rk.$$

We use the upper bounded computed from the chain of relations and the minimal element. In section 5.3 we will show that it is the optimal upper bound.

3. Stability of $\pi_1(Symp(X,\omega))$ in the c_1 positive cone

We prove the stability of $\pi_1(Symp(X,\omega))$ in P_K or its reduced counterpart. In particular, we have the following

Proposition 3.1. *On each simplical facet of P_K, the $\pi_1(Symp(X,\omega))$ is invariant.*

Here by a simplical facet, we mean a maximal simplex, which is relatively open, and of the same codimension in P_K.

To prove this, recall that in [LLWxi], we have the following stratification of $\mathcal{A}_w$, which is the space of almost complex structures compatible with some symplectic form that is isotopic to ω.

We have the following:

Definition 3.2. *Given $J\in\mathcal{A}_\omega$ and an exceptional class E, one may take a symplectic form ω' such that $J\in\mathcal{J}_{\omega'}$. There is a decomposition*

$$\mathcal{A}_\omega = \mathcal{A}^0_\omega(E)\sqcup\mathcal{A}^2_\omega(E)\sqcup\mathcal{A}^4_\omega(E)$$

such that $J\in\mathcal{A}^\dagger_\omega(E)\iff J\in\mathcal{J}^\dagger_{\omega',E}$, where $\dagger=0,2,4$.

Moreover, it is straightforward to introduce the relative to D, a (-2) symplectic sphere class version, of the above spaces and stratification.

Definition 3.3. *Let D be an ω-symplectic (-2) sphere class and $\mathcal{A}^D_\omega$ be the subspaces of $J\in\mathcal{A}_\omega$ such that there is a J-holomorphic embedded curve in class D. Then we have*

$$\mathcal{A}^D_\omega = \mathcal{A}^{D,0}_\omega(E)\sqcup\mathcal{A}^{D,2}_\omega(E)\sqcup\mathcal{A}^{D,4}_\omega(E)$$

such that $J\in\mathcal{A}^{D,\dagger}_\omega(E)\iff J\in\mathcal{J}^{D,\dagger}_{\omega',E}$, where $\dagger=0,2,4$.

Also, recall Proposition 3.28 in [LLWxi]:

Lemma 3.4. *For a form $u = (1|c_1, \cdots, c_{n-m}, c_{n-m+1}, \cdots, c_n) \in P^n_K$, here P^k_K is c_1 positive reduced symplectic cone, $c_{n-m} > c_{n-m+1} = \cdots = c_n$, and c_n is the minimal exceptional size. We define two types of lines:* **minimal exceptional line***: $\overline{uu_0}$ where $u_0 := (1|c_1, \cdots, c_{n-m}, 0, \cdots, 0)$,* **A-extremal line***: $\overline{uA}$, where $A := (1|1, 0, \cdots\cdots, 0)$. .*

For any $u_t \in L$ where L is the interior of $\overline{uu_0}, \overline{uA}$, we always have $\mathcal{A}_u \subset \mathcal{A}_{u_t}$.

We are going to prove Proposition 3.1 using the same kind of deformation as in Lemma 3.4:

Proof. Firstly, note that any two points u, u' can be connected by a path using two types of lines in Lemma 3.4, if they are in the simplicial facet of P_K. For each target form u', choose a decomposition in 2.3 and write $u' = \sum E_i$.

Then by Lemma 3.10 in [LLWxi], the strata $\mathcal{A}^2_\omega(E)$ in Definition 3.2 is covered by the unions of all $\mathcal{A}^D_\omega$'s where $D \cdot E_i \leq 0$ for some E_i in $u' = \sum E_i$.

Note it follows from Lemma 2.4 and Lemma 3.3 that the open part of $\mathcal{A}^D_\omega$ is invariant and the complement is of real codim at least 2 in $\mathcal{A}^D_\omega$ (this means codimension at least 4 in $\mathcal{A}_\omega$).

Also, note that

Lemma 3.5. *In the simplicial facet of P_K, the possible homology classes D are the same.*

Proof. By the root description of (-2) classes, Lemma 2.10 in [LLWxi], one only needs to check the simple roots are the same for the reduced c_1 positive cone. □

Hence we have double inclusion for the top strata of each $\mathcal{A}^D_\omega$. By Lemma 3.5, the top strata and codimension 2 strata have a part that is invariant under the deformation between u and u'; further, the complement of this invariant part has codimension 4 or higher in $\mathcal{A}_\omega$. Then the $\pi_i, i \leq 2$ of $\mathcal{A}_u$ and $\mathcal{A}_{u'}$ are the same.

Hence the stability Proposition 3.1 follows from this and the Kroheimer fibration.

□

4. Stratification of $\mathcal{J}_\omega$ according to $(H - E_1)$-fiber/section classes

We are going to give a lower bound for type $\mathbb{A}$ forms using $H - E_1$-fiber/section classes and their induced stratification on $\mathcal{J}_\omega$

4.1. $(H - E_1)$-fiber/section classes.
Let us recall a special type of sphere class from [LLW a] called $(H - E_1)$-fiber/section classes.

Definition 4.1. *Let X be a rational 4-manifold with canonical class $K = -3H + E_1 + \cdots + E_n$, and $A \in H_2(X, \mathbb{Z})$ such that $g_J(A) = (K \cdot A + A \cdot A)/2 = 0$. We call A an $(H - E_1)$-fiber type class if $A \cdot (H - E_1) = 0$, and A an $(H - E_1)$-section type class if $A \cdot (H - E_1) = 1$.*

Those classes have the nice property that their Gromov limits only have embedded components. This allows us to find a nice stratification of $\mathcal{J}_\omega$ and to apply Alexander duality. Here $\mathcal{J}_w$ is the space of almost complex structures compatible with ω.

Let us also recall from [LL20] that the symplectic sphere classes $\mathcal{S}_\omega \subset H_2(M,\mathbb{Z})$ and its subset $\mathcal{S}^{<0}$are going to give rise to a stratification of $\mathcal{J}_\omega$.

More precisely, we can decompose $\mathcal{J}_\omega$ into

Definition 4.2. *The* **prime subsets**

$$\mathcal{J}_\mathcal{C} := \{J \in \mathcal{J}_\omega | A \in \mathcal{S}^{<0} \textit{ has a } J\textit{-hol embedded representative iff } A \in \mathcal{C}\},$$

where $\mathcal{C}$ is a subset of spherical classes of $H_2(X)$ such that any pair of classes intersect positively.

Note that in general $\mathcal{J}_\mathcal{C}$ is a Banach(Fréchet) analytic subset of $\mathcal{J}_\omega$ of finite codimension. We also simply denote $Cod(\mathcal{C})$ as the codimension of $\mathcal{J}_\mathcal{C}$ in $\mathcal{J}_\omega$.

Also, recall the following fact about the stable curve of such classes

Proposition 4.3. *Let (M,ω) be a symplectic rational surface and $A \in \mathcal{S}_\omega$. Suppose ω is a reduced symplectic form and $A \cdot (H - E_1) = 0, 1$. If $A \in \mathcal{S}^{<0}$ and $A = \sum r_i C_i, r_i \in \mathbb{Z}^+$ be a homology decomposition from a Gromov limit, then each C_i is an embedded sphere class and $\mathcal{C} := \{C_i\}$ has $Cod(\mathcal{C}) > Cod(A)$.*

We give the following change of basis in $H^2(X,\mathbb{Z})$ in preparation for the D_{n-1} case and the inductive step. Note that $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \geq 1$ can be naturally identified with $\mathbb{C}P^2 \# (k+1)\overline{\mathbb{C}P^2}$. Denote the basis of H_2 by $B, F, E_1', \cdots, E_k'$ and $H, E_1, \cdots, E_k, E_{k+1}$ respectively. Then the transition on the basis is explicitly given by

$$\begin{aligned} B &= H - E_2, \\ F &= H - E_1, \\ E_1' &= H - E_1 - E_2, \\ E_i' &= E_{i+1}, \forall i \geq 2, \end{aligned} \tag{4}$$

with the inverse transition given by:

$$\begin{aligned} H &= B + F - E_1', \\ E_1 &= B - E_1', \\ E_2 &= F - E_1', \\ E_j &= E_{j-1}', \forall j > 2. \end{aligned} \tag{5}$$

$\nu H - c_1 E_1 - c_2 E_2 - \cdots - c_k E_k = \mu B + F - a_1 E_1' - a_2 E_2' - \cdots - a_{k-1} E_{k-1}'$ if and only if

$$\mu = (\nu - c_2)/(\nu - c_1), a_1 = (\nu - c_1 - c_2)/(\nu - c_1), a_2 = c_3/(\nu - c_1), \cdots, a_{k-1} = c_k/(\nu - c_1). \tag{6}$$

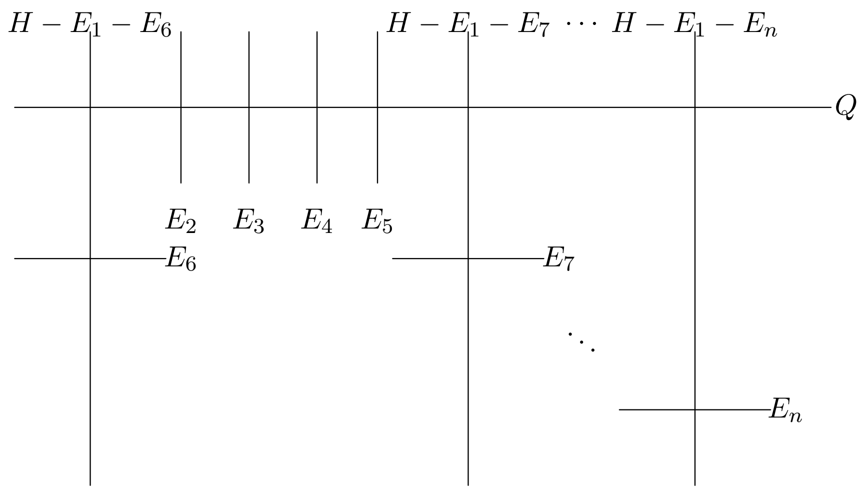
4.2. $Symp(M,\omega)$ **via stratification of** $\mathcal{J}_\omega$**.** We use the same strategy as in [LLW22]. We will generalize Lemma 5.33 and Theorem 5.35 in [LLW22], which together give the precise rank of $\pi_1(Symp_h(X,\omega))$ in the case of type $\mathbb{A}$:

Theorem 4.4. *Suppose the symplectic form is of type $\mathbb{A}$ on X_n, then we can give a lower bound of the rank of $\pi_1(Symp(X_n,\omega))$, which equals the upper-bound given in Proposition 2.8.*

In particular, for a generic symplectic form (there's no -2 Lagrangian sphere) then rank of $\pi_1(Symp(X_n,\omega)) = n(n+1)/2$.

Proof. **The upper bound** This is given by Proposition 2.8, and for a generic symplectic form, it is $1+2+\cdots+n = n(n+1)/2$. For an arbitrary type $\mathbb{A}$ form, one computes this according to the well-founded relation 2.6. We will show that this is the optimal upper bound.

Now we show that there is a lower bound of $\pi_1(Symp(X_n,\omega))$. Recall from [LLWxi] that we have the filling divisor



Where $Q = 2H - E_1 - \cdots - E_5$.

Complement $U = (\mathbb{C} - \{p_1, p_2, \cdots, p_{n-6}\}) \times \mathbb{C}$, and by Lemma 5.7 of [LLWxi], its compactly supported symplectomorphism group is connected.

And for a reduced form with $\omega = (1|c_1, \frac{1-c_1}{2}, \cdots, \frac{1-c_1}{2})$ and $c_1 > \frac{n-7}{n-3}$, we have

(7)
$$\begin{array}{ccccccc} Symp_c(U_{n-6}) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_h(X,\omega) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{Z}^{2n-7} & & (S^1)^{2n-6} \times \mathrm{Diff}^+(S^2, n-1) & & \mathcal{C}_0 \simeq \mathcal{S} \end{array}$$

When TSMC is connected, we consider the following portion of the LES:

$$\pi_2(\mathcal{J}_{top}) \to \mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \to 1 \tag{8}$$

Note that we can think of the following sequence

$$\pi_1(Symp_h(X,\omega))/\mathbb{Z} \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \to 1 \tag{9}$$

and its abelinization gives the following:

$$\mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to Ab[\pi_1(\mathcal{S})] \xrightarrow{f} Ab[\pi_0(Stab(C))] = \mathbb{Z}^{(n-1)(n-4)/2} \to 1 \tag{10}$$

In the next Proposition 4.6, we deal with $\pi_1(\mathcal{S})$, and show that its rank in $(n-1)^2+1:$

We need to find out the -2 curve that pairs at least one of the -1 curves negative in the configuration.

A complete list is the following:

Number of curves	Homology class
4	$H-E_i-E_j-E_k, \quad i,j,k \in \{2,\cdots,5\}$
$(n-2)(n-1)/2$	$E_s-E_t, \quad s,t\in\{2,\cdots,n\};$
$(n-1)(n-2)/2$	$H-E_1-E_p-E_q, \quad p,q\in\{2,\cdots,n\}.$
$n-5$	$2H-E_1-\cdots E_5-E_i, i>5.$

TABLE 1. List of curves

Hence the total number is $(n-1)(n-2)+n-1=(n-1)^2+1.$

□

Further, we have the following decomposition of such classes of $(h-e_1)$-fiber/section classes, proved in [LLWxi] Section 5:

Lemma 4.5. *There are three possible types of bubblings of the above list of homology classes if it is not an embedded curve:*

- $(2h-e_1-e_2-e_3-e_4-e_5-\sum_{i\in I_0} e_i)+\sum_{i\in I_0} e_i$;
- $(h-\sum_{i_0\in I_0} e_{i_0})+(h-e_1-\sum_{i_1\in I_1} e_{i_1})+\sum_{i\in I_2} e_i$;
- $(e_1-\sum_{i_0\in I_0} e_{i_0})+(h-e_1-\sum_{i_1\in I_1} e_{i_1})+(h-e_1-\sum_{i_2\in I_2} e_{i_2})+\sum_{i_3\in I_3} e_{i_3}$.

Proposition 4.6. *The rank of $\pi_1(\mathcal{S})$ is $(n-1)(n-2)+n-1=(n-1)^2+1$, generated by the above -2 symplectic sphere classes.*

Proof. In equation (8), we want to compute $\pi_1(S)$ from H_1 of the complement of Codimension at least 2 strata of $\mathcal{J}_\omega$ and Alexander duality. Let us recall the following:

Theorem 4.7. *Let $\mathcal{X}$ be a Hausdorff space, $\mathcal{Z}\subset\mathcal{Y}$ a closed subset of $\mathcal{X}$ such that $\mathcal{X}-\mathcal{Z}, \mathcal{Y}-\mathcal{Z}$ are paracompact manifolds modeled by topological linear spaces. Suppose $\mathcal{Y}-\mathcal{Z}$ is a closed co-oriented submanifold of $\mathcal{X}-\mathcal{Z}$ of codimension p, then we say $(\mathcal{Y},\mathcal{Z})$ is a closed relative submanifold of $(\mathcal{X},\mathcal{Z})$ of codimension p. By relative version of Alexander-Pontrjagin duality* *in [Eel61] when taking constant coefficient, we have an isomorphism $H^i(\mathcal{X}-\mathcal{Z},\mathcal{X}-\mathcal{Y};G)\cong H^{i-p}(\mathcal{Y}-\mathcal{Z};G)$ for any Abelian coefficient group G.*

Let $\mathcal{X}$ be the space of $J\in\mathcal{J}_\omega$ where all irreducible pseudo-holomorphic curves have non-negative H coefficients. This means the complement $\mathcal{J}_\omega-\mathcal{X}$ is the space of J such that there exists an irreducible J-holomorphic curve representing class A such that $A\cdot H<0$.

Lemma 4.8. *If ω is reduced, $\mathcal{J}_\omega-\mathcal{X}$ has an embedded J-holomorphic sphere representing class $-kH+(k+1)E_1-\sum_i E_i$. $\mathcal{X}$ has the same (trivial) π_1 as $\mathcal{J}_\omega$. Furthermore, $\mathcal{J}_\omega-\mathcal{X}$ is closed in $\mathcal{J}_\omega$, and hence $\mathcal{X}$ is a manifold.*

Proof. The first statement is a combination of Lemma 4.1 in [Zha17] and Lemma 3.2 in [Che20]. Lemma 4.1 in [Zha17] says that for $M=\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}$, $C=aH+\sum b_iE_i$ with $a<0$ is represented by an irreducible curve, then $C=-nH+(n+1)E_1-\sum_{k_j\neq 1}E_{k_j}, n\geq 1$ up to a Cremona tranform. Then by Lemma 3.2 in [Che20], for a reduced symplectic form, any such class is an embedded sphere class, and we don't need the Cremona transform.

For the second statement, note that $\mathcal{J}_\omega-\mathcal{X}$ is a union of submanifolds characterized by the existence of embedded spheres of self-intersection less than -2. Hence those submanifolds each have real codimension at least 4 in $\mathcal{J}_\omega$. By transversality, the complement $\mathcal{X}$ has the same fundamental group as $\mathcal{J}_\omega$, i.e. it is simply connected.

Now we prove the third statement. Assume for contradiction, $\mathcal{J}_\omega-\mathcal{X}$ is not closed in $\mathcal{J}_\omega$. This is to say that $\overline{\mathcal{J}_\omega-\mathcal{X}}\cap\mathcal{X}\neq\emptyset$. Hence by Gromov compactness theorem, there is some class A with $A\cdot H<0$, that has a stable curve $A=\sum_i r_iS_i, r_i>0$ which is J-holomorphic for some J in $\mathcal{X}$. By definition, any irreducible curve S_i that is holomorphic w.r.t this J has a non-negative coefficient on H. This is a contradiction against $A\cdot H=(\sum_i r_iS_i)\cdot H<0$. Hence $\mathcal{J}_\omega-\mathcal{X}$ is closed in $\mathcal{J}_\omega$. Because $\mathcal{J}_\omega$ is itself a manifold, removing a closed subset we get $\mathcal{X}$, which is also a manifold.

□

We then define the following: $\mathcal{Y}\subset\mathcal{X}=\{J$ s.t. some classes in configuration 4.2 is not embedded $\}$. $Z\subset Y=\{J$ s.t. curves in the configuration 4.2 has a stable representative and the total codimension of the codimension of the components is higher than $2\}$.

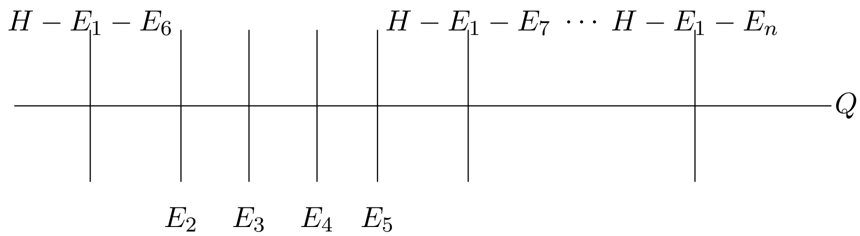
Moreover, a complete list has been given in table 1. Notice that those are either $(H-E_1)$-fiber classes or $(H-E_1)$-section classes. Recall that in the stable curve of such classes, each component is an embedded rational curve and the codimension of the stable curve is no less than that of the class itself.

Now let us prove the following Lemma:

Lemma 4.9. *$\mathcal{Z}$ and $\mathcal{Y}$ are closed subsets in $\mathcal{X}$. Consequently, $\mathcal{X}-\mathcal{Z}$ is an open submanifold of $\mathcal{X}$ and $\mathcal{X}-\mathcal{Y}$ is a closed submanifold of $\mathcal{X}-\mathcal{Z}$.*

Proof. First, we deal with the closedness of $\mathcal{Y}$. Assume the contrary $\mathcal{Y}$ is not closed. This means that there is a sequence of $J_n\in\mathcal{Y}$, each admits an embedded J_n-holomorphic (-2) sphere, converging to $J_0\in\mathcal{X}-\mathcal{Y}$ which has no embedded (-2) spheres. This is a contradiction.

Similarly, assume that $\mathcal{Z}$ is not closed, the same contradiction will occur. □



Hence we have $\mathcal{X} - \mathcal{Y}$ and $\mathcal{X} - \mathcal{Z}$ are both submanifolds of $\mathcal{X}$ and they are also submanifolds of $\mathcal{J}_\omega$.

Note that it also immediately follows that $\mathcal{Y} - \mathcal{Z}$ is closed in $\mathcal{X} - \mathcal{Z}$. This is essential for the Alexander duality. Notice that by equations (4.5), when there are no irreducible curves with negative H coefficients, each component of a stable curve of any class in 4.2 is an embedded sphere of non-positive self-intersection.

Now Alexander duality can be applied to the pair $\mathcal{X} - \mathcal{Y}$ and $\mathcal{Y} - \mathcal{Z}$. Note that $\mathcal{S}$ is exactly $\mathcal{X} - \mathcal{Y}$, whose H_1 can be computed by counting the number of (connected components) of codimension 2 strata. Notice that for a generic symplectic form, the transitivity of $Symp(M, \omega)$ acting on -2 symplectic spheres and the connectivity of $Symp(M, \omega)$ altogether yield that the number of connected cod=2 strata is the number of -2 curves listed as above. $\square$

5. Optimal upper bound and $\pi_1(Ham(M, \omega))$ in the type $\mathbb{D}$ case

In this section, we prove that the well founded-relation in 2.6 gives us the optimal lower bound. Based on this, we show it agrees with the lower bound given in the previous section. Moreover, in the type $\mathbb{D}$ case, we explicitly show that $\pi_1(Ham(M, \omega))$ is the free abelian group of the expected rank.

5.1. **The equal small size blowup.** First we detail how the upper bound behaves under the induction step in the well founded relation:

Proposition 5.1. *If we compute the upper bound using a known rank of $\pi_1(Symp(X_{r-1}, \omega))$ for an ω on X_{r-1} with $b_2 = r$, then after blowing up k equal size small balls, the lower bound of rank $\pi_1(Symp)$ is equal to the upper bound.*

We are going to work with the following configuration of curves

Notice that by the base change, we can regard X_n as $n - 1$ point blow-up of $S^2 \times S^2$ or F_1. This configuration of curves is a collection of the section class and n-1 fiber classes. Hence the complement is diffeomorphic to $\mathbb{C}^2$. When the blowup sizes are small, the complement is convex as a symplectic domain.

We can also write down the following diagram:

(11)
$$\begin{tikzcd}
Symp_c(\mathbb{C}^2, \omega_{std})) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X, \omega) \arrow[d] \\
 & \mathbb{Z}^{n-2} & (S^1)^{n-1} \times \mathrm{Diff}^+(S^2, n-1) & \mathcal{C}_0 \simeq \mathcal{J}_C
\end{tikzcd}$$

Now let us write down the following long exact sequence:

$$\cdots \to \mathbb{Z} \to \pi_1(Symp_h(X_m,\omega)) \to \pi_1(\mathcal{S}) \xrightarrow{\phi} \pi_0(\mathrm{Diff}^+(S^2, m-1)) \xrightarrow{\psi} \pi_0(Symp_h(X_m,\omega)) \to 0 \tag{12}$$

Let us also note the following abelian version of the sequence:

$$\mathbb{Z} \to \pi_1(Symp_h(X_m,\omega)) \to H_1(\mathcal{S}) \xrightarrow{f} Ab(Im(\phi)) \to 0. \tag{13}$$

Then let us prove the following Lemma:

Lemma 5.2. *The $\mathbb{Z}$ factor injects into $\pi_1(Symp_h(X_m,\omega))$ for*

- *a toric symplectic form which admits a divisor in Figure 5.1;*
- *or any symplectic form that is close enough to point A in the normalized reduced cone (notice that this form can only be of type $\mathbb{A}$ or $\mathbb{D}$).*

Proof. Let us first deal with the toric case. By assumption, we can assume that the divisor in Figure 5.1 is part of the toric boundary. Then by [KKP15], every circle action on a symplectic toric 4-manifold can be extended to a torus action. Hence the circle action from $Stab(C)$ can be realized as a factor of the torus action. By Theorem 1.3 and 1.25 of [MT10], a torus action injects on to $\pi_1(Symp)$. Hence we have the injective map from $\mathbb{Z}$ to $\pi_1(Symp)$, for a toric symplectic form which fixes the given divisor.

Now we deal with the second case when a symplectic form is close enough to A in the reduced symplectic cone. First note that this form admits a circle action when the section area is large enough. This is because there is a circle action on $S^2 \times S^2$ or $F(1)$ which rotates each of the fibers. There is a smooth circle action after the blowup, which still rotates the generic smooth fiber and rotates each component of the singular fibers. When the section area is large enough, this smooth circle action can be made Hamiltonian because of the weight of the Karshon graph. More precisely, we start with $S^2 \times S^2$ where the base area is at least $4n$ of the area of the fiber (or $F(1)$ where the area of the positive section is at least $4n+2$ of the fiber area). Then start with the circle action of the highest weight on $S^2 \times S^2$ ($F(1)$ respectively), by Lemma 3.2 in [KK07] conditions (1)(2) and (3), we can inductively blowup $k, 1 \le k \le n$ ($1 \le k \le n+1$ respectively) points of weight at most half of the size of the fiber class. Note that the condition $4n$ on $S^2 \times S^2$ (or $4n+1$ on $F(1)$) are sufficient conditions, and they are equivalent to the form class being close enough to the point A in the normalized reduced cone. Also, the circle action agrees with the smooth action which fixes rotates the generic smooth fiber and rotates each component of the singular fibers. Now we find that the circle in $Stab(C)$ can be realized as such a circle action, which is obtained from the equivariant blowup of the circle action on $S^2 \times S^2$ or $F(1)$. Also, note that the (minimal) circle action injects into $\pi_1(Symp(X,\omega))$ when $X = S^2 \times S^2$ or $F(1)$. We claim that the blowup circle action also injects into $\pi_1(Symp(\tilde{X},\tilde{\omega}))$. The reason is the following: in the argument of Corollary 6.4 in [McD08], the counting of the rank of $\pi_1(Symp(\tilde{X},\tilde{\omega}))$ is guided by the Hamiltonian bundle structure. The Hamiltonian bundle on $\tilde{X}$ is spanned by the blowup of the original Hamiltonian bundle on X and the exceptional classes. Notice that the original Hamiltonian bundle corresponds to the circle action and the blowup Hamiltonian bundle corresponds to the blowup circle action. Hence we have the desired injection from $\mathbb{Z}$ to $\pi_1(Symp)$.

□

5.2. **Type $\mathbb{D}$ forms on $\mathbb{C}\mathbf{P}^2\#\mathbf{n}\overline{\mathbb{C}\mathbf{P}^2}$.** For the convenience of computation, we divide all type $\mathbb{D}$ forms into 2 families:

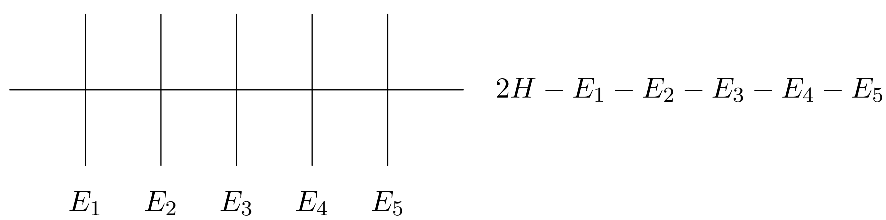
- MA or $MAL_{i_1}\cdots L_{i_k}$.
- ME or $MEL_{i_1}\cdots L_{i_k}$.

Notice that when the MA and ME cases are done, every other case can be computed using Proposition 5.1.

5.2.1. *MA or $MAL_{i_1}\cdots L_{i_k}$ cases.*

Proposition 5.3. *The rank of $\pi_1(Symp_h(X,\omega))$ for the $MA, MAE, \cdots,$ cases of a $n-$points blow up:*
• *if $\omega \in MA$, then $rk(\pi_1) = n$, and in particular, we know it is free abelian, i.e. $\pi_1(Ham) = \mathbb{Z}^n$.*
• *If ω lies on any other lines, we can always find a lower bound such that it equals the upper bound given by [McD08] and Proposition 3.21 in [LLW22].*

5.2.2. *ME or $MEL_{i_1}\cdots L_{i_k}$ cases.* We start with the ME on a 6-point blowup here. Consider the following configuration together with E_6.



The complement $\mathbb{T}^*\mathbb{R}P^2$ has $Symp_c$ weakly homotopic to $\mathbb{Z}$. The homotopy LESs of the diagram will have an S^1 in the π_1 level of the fiber, and every higher π_i being trivial.

For the space of configuration, its fundamental group is generated by 5 curves $E_1 - E_6, \cdots, E_5 - E_6, 2H - E_1 - \cdots - E_5 - E_6$. Then at least, the fundamental group has rank 6.

On the other hand, by blowing down to the monotone 5-point blowup, the upper bound of the rank of the fundamental group is 6. Hence the rank is precisely 6.

For more points blow up, we will consider the above configuration together with $E_6, E_7, \cdots, E_n$. A similar computation will give us that the rank is precisely the number of symplectic classes among $E_1 - E_i, \cdots, E_5 - E_i, 2H - E_1 - \cdots - E_5 - E_i, E_i - E_j$, where $i, j \geq 6$.

Combining the above discussion, we have the following theorem:

Proposition 5.4. *For a rational surface whose Lagrangian system is of type $\mathbb{A}$ or $\mathbb{D}$ in the c_1-positive symplectic cone,*

$$Q = PR[\pi_0(Symp_h(X,\omega))] + Rank[\pi_1(Symp(X,\omega))] - rank[\pi_0(Symp_h(X,\omega))]$$

is a constant only depending on the topology. In particular, $Q = 1+2+\cdots n$ for the n points blow up of $\mathbb{C}P^2$.

5.3. **Proof of the Main theorem and applications.** The well-founded relation allows us to do well-founded induction to prove results about Rank of $\pi_1(Ham(M,\omega))$. Recall that to do well-founded induction we only need to prove the minimal case and the induction step.

There has been a complete understanding of $\pi_1(Ham(X_n,\omega))$ for any symplectic forms when $n \leq 5$, see [LL20] and [LLW22]. In this section, we use those results to completely calculate $\pi_1(Ham(X_n,\omega))$ of any X_n for type $\mathbb{A}$ or $\mathbb{D}$ in the c_1-positive symplectic cone.

Note that we need the minimal case D_{n-1} on X_n. Then we need to prove that if u satisfies statement $P(u)$, then inductive steps $u \prec v$ give the statement $P(v)$.

Proof of Proposition 5.1

Proof. Then for the upper bound, we have $R_r + rk$. For the lower bound, we have

$$rank(\pi_1(\mathcal{S}_n)) - rank(Ab(PB_{n-1})) + 1 + rank(\pi_0 Symp(X_n,\omega_n)).$$

Now we try to relate $rank(\pi_1(\mathcal{S}))$ with R_r.

Notice that we assume R_r agrees with the lower bound. This means that we have

$$R_r = rank(\pi_1(\mathcal{S}_r)) - rank(Ab(PB_{r-1})) + 1 + rank(\pi_0 Symp(X_r,\omega_r)). \tag{14}$$

Now we compare $rank(\pi_1(\mathcal{S}_n))$ with $rank(\pi_1(\mathcal{S}_r))$. There are $(r-2)k$ new classes of curves with $E_s - E_t$, where $1 \leq s \leq r, r < t \leq n$. There are $(r-2)k$ new classes of curves with $H - E_1 - E_i - E_j$, where $1 < i \leq r, r < j \leq n$. There are $(k-1)k/2$ new classes of curves with $H - E_1 - E_u - E_v$, where $r < u, v \leq n$. Finally, there are k new classes of curves with $2H - E_1 - \cdots - E_5 - E_w$, where $r < w \leq n$. Hence the difference between $rank(\pi_1(\mathcal{S}_n))$ and $rank(\pi_1(\mathcal{S}_r))$ is

$$(r-2)k + (r-2)k + (k-1)k/2 + k = \frac{1}{2}k^2 + 2rk - \frac{7}{2}k.$$

Then we compare the upper bound with the lower bound and compute their difference:

$$(R_k + kr) - (rank(\pi_1(\mathcal{S}_n)) - rank(Ab(PB_{n-1})) + 1 + rank(\pi_0 Symp(X_n,\omega_n))).$$

Plug in equation (14), we have

$$\begin{aligned} &rank(\pi_1(\mathcal{S}_r)) - rank(Ab(PB_{r-1})) + 1 + rank(\pi_0 Symp(X_r,\omega_r)) + kr \\ &-(rank(\pi_1(\mathcal{S}_n)) - rank(Ab(PB_{n-1})) + 1 + rank(\pi_0 Symp(X_n,\omega_n))). \end{aligned} \tag{15}$$

Notice that $\pi_0 Symp(X_n,\omega_n)) = \pi_0 Symp(X_r,\omega_r))$, by [LLWxi] Proposition 3.5.

Now this is

$$-(\frac{1}{2}k^2 + 2rk - \frac{7}{2}k) + (k+r-2)(k+r-5)/2 - (r-2)(r-5)/2 + kr = 0.$$

Hence we completed the proof. □

6. Applications and Discussions

We shall apply the stability results and computation of $\pi_1(Symp)$ to the space of ball packings and circle actions.

6.1. On space of embeddings of balls. Let us prove that the space of embeddings also has homotopy stability in the π_1 level when the sizes of the ball satisfy the c_1 positive condition when blowing up. We still focus on the minimal blowing down process.

Let M be a rational 4-manifold, ω on M a c_1 positive symplectic form. Let $[\omega]$ be $(1|m_1,\cdots,m_k)$ and we write $\vec{(c)}$ be the n-k dimensional vector $(c,\cdots,c)$. We further assume that the blow up symplectic class $(1|m_1,\cdots,m_k,c,\cdot,c)$ is also c_1 positive on $M\#(n-k)\overline{\mathbb{C}P^2}$.

By [LP04, Theorem 2.5(ii)], we have the following fibration

$$Symp(M,\sqcup_i B_i(c);\omega)^{U(2)} \to Symp(M,\omega) \to \mathrm{Emb}^*_\omega(B^4(\vec{c}),M), \tag{16}$$

where $\mathrm{Emb}^*_\omega(B^4(\vec{c}),M)$ is the space of (ordered, parametrized) embeddings from m disjoint balls of capacity c to M.

Theorem 6.1. (1) *The π_1 of $Emb^*_\omega(B^4(\vec{c}),M)$ is in the stable chamber of $\pi_1(Symp)$ as in Proposition 3.1.*

(2) *Notation as above, $Emb^*_\omega(B^4(\vec{c}),M)$ is simply connected if $c<m_k$.*

Proof. We then have the following portion of the commutative diagram of long exact sequence, induced by the inclusion map $\mathrm{Emb}^*_\omega(B^4(\vec{\delta}),M) \xrightarrow{\phi_\delta} \mathrm{Emb}^*_\omega(B^4(\vec{c}),M)$.

$$\begin{tikzcd}
\pi_1(Symp_h(\widetilde{M},\omega_{\vec{c}})) \arrow[r, "\phi^1_\delta"] \arrow[d, "i_*"'] & \pi_1(Symp_h(\widetilde{M},\omega_{\vec{\delta}})) \arrow[d] \\
\pi_1(Symp_h(M,\omega)) \arrow[r, "\cong"] \arrow[d] & \pi_1(Symp_h(M,\omega)) \arrow[d] \\
\pi_1(\mathrm{Emb}^*_\omega(B^4(\vec{c}),M) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \pi_1(\mathrm{Emb}^*_\omega(B^4(\vec{\delta}),M) \arrow[d, "\beta_\delta"'] \\
\pi_0(Symp_h(\widetilde{M},\omega_{\vec{c}})) \arrow[r, "\phi^0_\delta"] \arrow[d, "i_*"'] & \pi_0(Symp_h(\widetilde{M},\omega_{\vec{\delta}})) \arrow[d] \\
\pi_0(Symp_h(M,\omega)) \arrow[r, "\cong"] & \pi_0(Symp_h(M,\omega))
\end{tikzcd} \tag{17}$$

Note that the maps ϕ^1_δ (by Proposition 3.1) and ϕ^0_δ (by [LLWxi] Corollary 3.28) are isomorphisms, when $c<m_k$. Then it follows from five-Lemma that the middle map α is also an isomorphism.

Then statement (1) follows from the fact the pi_0 and π_1 of $Symp$ have the same stability chamber for any X_k.

Now we prove statement (2): We shall compare the ball sizes of c with very small ball sizes δ. First it is straightforward to see that the space $\mathrm{Emb}^*_\omega(B^4(\vec{\delta}), M)$ is simply connected for small enough δ. The proof is similar to [LLW22] Theorem A.1. We start with $\mathrm{Emb}^*_\omega(B^4(\vec{0}), M)$, which is the configuration space, and it is simply connected Since M is itself simply connected. Then statement (2) follows from diagram (17) and five Lemma.

□

6.2. **A discussion on Hamiltonian loops vs Circle actions.** This subsection is a discussion on circle actions and Hamiltonian loops. Note that for all circle actions here we assume it is effective and Hamiltonian.

It is known to Kedra that a blowup of an algebraic surface of general type admits a Hamiltonian loop that is not represented by circle actions. In the rational surface case, [Anj+23] gave an example that some loops can be circle-representable but under deformation become non-representable by circle actions. Inspired by [Anj+23], and a private communication with Silvía Anjos, we have the following discussion on Circle actions vs Hamiltonian loops.

Example 6.2. *Consider $X_k, k \geq 5$, the symplectic form ω in class $[1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \epsilon, \cdots, \epsilon]$, $\epsilon << \frac{1}{3}$.*

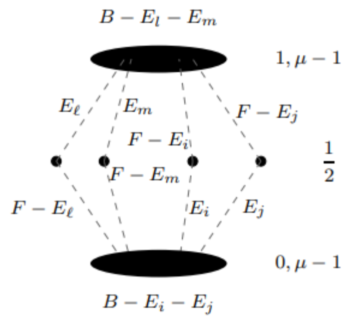
By Proposition 2.8 and Proposition 5.1, the rank of $\pi_1(Symp(X_k,\omega))$ is $4(k-4)$. Meanwhile, there is no circle action compatible with this symplectic form. The reason is that any circle action on the blowup can be equivariantly blown down(cf. [KKP15]). It is known to [HK19] and [KK07] that there is no circle action on the monotone 4-point blow (a for in class $[1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}]$ on X_4). We remark that after a base change (4), as a 3 blowup of size of $\frac{1}{2}$ from $\S^2 \times S^2$, with a symplectic form $\mu\sigma \oplus \sigma$. Note that $\mu = 1$ correspondence to $\frac{1}{3}$ blowup from $\mathbb{C}P^2$.

This case is similar to Kedra's example in the rational surface case.

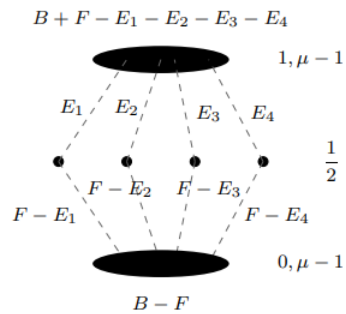
Example 6.3. *Let's recall the example of [Anj+23]: On X_5,, the symplectic form ω_a in class $[1, a, \frac{1-a}{2}, \frac{1-a}{2}, \cdots, \frac{1-a}{2}]$, $a \geq \frac{1}{3}$.*

By Theorem 1.3 of [LLW22], the rank of $\pi_1(Symp(X_k,\omega))$ is 5. In [Anj+23], it is shown that some of those Hamiltonian loops have circle actions, and some do not. Further, this changes as one deforms ω_a by varying a. Their argument is to analyze the possible Karshon graph (cf. [Kar99]) up to symplectomorphism, for different a. To simplify the discussion, they did a base change using (4).

One can always obtain the following types of Karshon graphs:

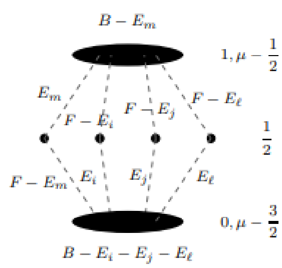


Circle action $z_{0,ij}$

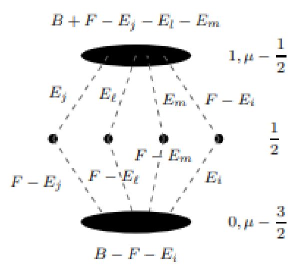


Circle action z_1

When $\mu > \frac{3}{2}$, which means $a > \frac{1}{2}$, one has the following two types of new Karshon graphs



Circle action $z_{0,ij\ell}$



Circle action $z_{1,i}$

When $1 < \mu \leq \frac{3}{2}$, which means $\frac{1}{3} < a < \frac{1}{2}$, there are 4 circle actions from Figure 6.3 and this means some Hamiltonian loops not represented by circle actions. When $\mu > \frac{3}{2}$ or equivalently $1 > a > \frac{1}{2}$, there are extra circle actions but they represent the same element in $\pi_1(Symp(X_5, \omega_a))$.

Note that this is a special case near the neighborhood of the monotone point $\omega_{mon} = [1, \frac{1}{3}, \cdots, \frac{1}{3}]$. In [Anj+23], it conjectures that there is a neighborhood of ω_{mon} in the symplectic cone such that there are not enough circle actions to represent all Hamiltonian loops.

Example 6.4. *This is a direct generalization of the previous example 6.3 and [Anj+23]. Let's consider X_6,, the symplectic form ω_a in class $[1, a, \frac{1-a}{2}, \frac{1-a}{2}, \cdots, \frac{1-a}{2}]$, $1 > a \geq \frac{1}{3}$.*

When $1 < \mu \leq \frac{3}{2}$, which means $\frac{1}{3} < a < \frac{1}{2}$, there are no circle actions. Because any Karshon graph needs to be a blowup from Figure 6.3, but any curve on Figure 6.3 has area less than $\frac{1}{2}$ and cannot be blown up. This means all (6, by Proposition 5.1) Hamiltonian loops are not represented by circle actions. When $\mu > \frac{3}{2}$ or equivalently $1 > a > \frac{1}{2}$, there are circle actions from blowing up the top or bottom curves in Figure 6.3, and they represent certain Hamiltonian loops.

Further, on $X_k, k \geq 6$, any symplectic form obtained by a ϵ blowup of ω_a on X_6 has the above property.

Observe that when $\mu > 2$, there are plenty of circle actions by blowing up the Karshon graphs in 6.3. It is an interesting question to check whether this forms a basis for $\pi_1(Symp(X_5, \omega_a))$, and the method of [Anj+23] applies here.

Lemma 6.5. *For mu $> \frac{k-3}{2}$, or equivalently $a > \frac{k-3}{k-1}$, there are Hamiltonian circle actions on X_k.*

Proof. When $mu > \frac{k-3}{2}$, there are Karshon graphs giving circle actions

- When k is odd, the bottom fixed curve is given by $B - \frac{k-3}{2}F$, and the top is given by $B + \frac{k-1}{2}F - E_1 - \cdots - E_{k-1}$.
- When k is even, the bottom fixed curve is given by $B - \frac{k-4}{2}F - E_1$, and the top is given by $B + \frac{k-2}{2}F - E_2 - \cdots - E_{k-1}$.

The vertical curves are given by E_i and $F - E_i$, making the Karshon graph a genus 0 Lefschetz fibration with $k-1$ isolated singular fibers. The circle action rotates the generic fiber in class F, and rotates each component of the singular fiber $F = E_i + (F - E_i)$. □

Remark 6.6. *It is not straightforward to show the converse of Lemma 6.5. The reason is there are abundant multiple section curves in class $pB + qF - \sum r_i E_i$, and it is not easy to exclude all possible Karshon garphs with those curves.*

Combine Lemma 6.5 and the stability results, we end this section with the following conjecture:

Conjecture 6.7. (1) *X_k with a symplectic form on MA has Hamiltonian circle actions if and only if $\mu > \frac{k-3}{2}$ or equivalently, $a > \frac{k-3}{k-1}$.*

(2) *Moreover, for X_k with a reduced symplectic form, that is c_1-nonpositive, or is c_1-small ($c_1 \cdot [\omega] = \epsilon$), any Hamiltonian loop cannot be represented by Hamiltonian circle actions.*

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