CHAMBERS IN THE SYMPLECTIC CONE AND STABILITY OF SYMP FOR RULED SURFACE

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Abstract. We continue the work of [9] to prove that for any non-minimal ruled surface (M,ω), the stability of π_0, π_1 of $Symp(M,\omega)$ is guided by embedded J-holomorphic curves. Further, we proved that for any fixed sizes blowups, when the area ratio μ between the section and fiber goes to infinity, there is a topological colimit of $Symp(M,\omega_\mu)$. In particular, when the blowup sizes are all equal to half of the area of the fiber class, there are non-trivial symplectic mapping classes in $Symp(M,\omega) \cap \mathrm{Diff}_0(M)$, which are not Dehn twists along Lagrangian spheres.

Contents

1. Introduction

In this note, we study some topological aspects of symplectomorphism groups, along the line of [1, 3, 29, 17, 4, 2, 7, 19, 21, 6], etc, We'll address the topological behavior of the symplectomorphism groups as the form ω_u varies within the symplectic cone. This is a follow-up note of [9], which addresses the symplectic stability and symplectic isotopy conjecture in a non-minimal irrational ruled surface setting. Recall that the conjecture informally states that the symplectic cone has chambers such that symplectomorphism groups have homotopy groups stable or invariant within the chambers.

The paper [9] established the symplectic stability conjecture for a one-point blow up, and it left out the discussion for more point blowups. As will be explained in section 3, the difficulty will be finding enough embedded J-holomorphic curves in given homology classes. This is the reason that we only partially establish this conjecture for more points blowup.

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More precisely, we will focus on the stability of π_0, π_1 of $Symp(X)$, where $X = M_g\#n\overline{\mathbb{C}P^2}$, while the symplectic form varies in a family such that $\omega_t(\Sigma_g) \to \infty$ and other symplectic areas remain the same.

Let $M_g = \Sigma_g \times S^2$. By McDuff's classification results [28], any symplectic form is diffeomorphic to $\mu\sigma_{\Sigma_g} \oplus \sigma_{S^2}$ for some $\mu > 0$, up to diffeomorphism and normalization. Such classification result also holds in the blowups $M_g\#n\overline{\mathbb{C}P^2}$ [24]: if one picks up ω on $M_g\#n\overline{\mathbb{C}P^2}$, then after normalization ω has areas $(\mu, 1, e_1, \cdots, e_n)$ on the homology classes $B, F, E_1, \cdots, E_n$, where $\mu > 0, e_1 + e_2 < 1, 0 < e_i < 1$, $e_1 \geq e_2 \geq \cdots \geq e_n, e_1 < \mu$, choosing the standard basis $B, F, E_1, \cdots, E_n$ and associate coefficients $(\mu, 1, e_1, \cdots, e_n)$ to get a cohomology class, then the symplectic forms in this cohomology class are isotopic. cf. [27, 25]. After normalization, the vector $u = (\mu, 1, e_1, \cdots, e_n)$ determines all possible symplectic form cohomology classes and belongs to a convex region Δ^{n+1} in $\mathbb{R}^{n+1}$, whose boundary walls are n-dimensional convex regions given by linear equations. We will be concerned with symplectic deformations inside this region Δ^{n+1} for the n-points blowups.

Here is a mega conjecture about how the deformation of ω changes the topology of $Symp(M, \omega)$:

Conjecture 1.1. *Let (M, ω) be a symplectic 4-manifold. We partition the symplectic cone of M as above, if possible. If ω_1 and ω_2 belongs to the same chamber of the symplectic cone of M, then $\pi_i(Symp(M, \omega_1)) = \pi_i(Symp(M, \omega_2)), \forall i \geqslant 1$.*

Recall that in [9], Conjecture 1.1 is proved for one point blowup of ruled surface. Here we prove the conjecture for π_0, π_1 of $Symp(M, \omega)$ for other non-mimimal ruled surfaces:

Theorem 1.2. *Let M be $\Sigma_g \times S^2\#k\overline{\mathbb{C}P^2}$ Suppose $\mu_i > g, i = 1, 2$ for $[\omega_1] = [\mu, 1, c_1, \cdots, c_n]$, $[w_2] = [\mu + \delta, 1, c_1, \cdots, c_n], \delta > 0$, then the groups π_0 and π_1 of $Symp(M, \omega_1)$ and $Symp(M, \omega_2)$ are the same.*

This also grants that there is a topological colimit when $\mu \to \infty$.

And in the following case, we can fully establish the stability conjecture.

Theorem 1.3. *The homotopy type of $G^g_{\mu,n}$ is constant for $\frac{k}{2} \leq \mu \leq \frac{k+1}{2}$, for any integer $k \geq 2g$. Moreover as μ passes the half integer $\frac{k+1}{2}$, all the groups $\pi_i, i = 0, \cdots, 2k + 2g - 1$ do not change.*

Moreover, in the special case when the blowup sizes are all equal to half of the area of the fiber $[S^2]$, we have the following theorem on the disconnectedness of the topological colimit $\mathcal{D}^n_g$. For a more detailed description of $\mathcal{D}^n_g$, see Definition 4.5.

Proposition 1.4. *Take $M_g\#n\overline{\mathbb{C}P^2}$ with a form in the class $[\mu, 1, \frac{1}{2}, \cdots, \frac{1}{2}]$. Then let μ go to ∞.*

(1) *$\mathcal{D}^n_g$ is weakly homotopic to $G^n_{\infty,g}$.*
(2) *The group $\mathcal{D}^n_g$ is disconnected when $g \geq 2$.*
(3) *When $\mu \to \infty$, s.t. for $i = 0, 1$, $\pi_i(G^n_{u,g}) = \pi_i(G^n_{\infty,g})$ for $i \leq min\{Cod(u)\} - 1$, and hence the groups $G^n_{u,g}$ are disconnected for $g \geq 2$.*

Notice that the above theorem extends the results of [9] from the one-point blowup of a minimal ruled surface to arbitrary points blowups. It remains open what the group $\pi_0 D_g^n$ is, and whether Proposition 1.4 holds for other symplectic forms. We hope to explore these questions in a future work.

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2. Preliminary

By [25], if M is a closed, oriented 4-manifold with $b^+ = 1$, the symplectic canonical class is unique once we fix $u = [\omega]$, and we denote it by K_u.

Let $\mathcal{E}$ be the set of exceptional sphere classes, and $\mathcal{C}_M$ denotes the symplectic cone.

In Theorem 4 of [25], Li-Liu showed that if M is a closed, oriented 4-manifold with $b^+ = 1$ and if the symplectic cone $\mathcal{C}_M$ is nonempty, then

$$\mathcal{C}_M = \{e \in P | 0 < |e \cdot E| \quad \text{for all} \quad E \in \mathcal{E}\}.$$

Note that the way we partition the normalized symplectic cone is by looking at the homology classes of potential symplectic curves. To that end, we will now fix some notation:

Definition 2.1. *Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic curves and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,*

$$K_\omega \cdot A = -A \cdot A - 2 + 2g(A). \tag{1}$$

For each $A \in \mathcal{S}_\omega$ we associate the integer

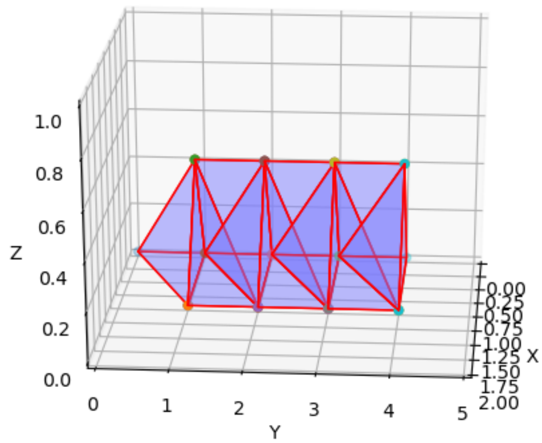
$$cod_A = 2(-A \cdot A - 1 + g).$$

We can define $\mathcal{S}_u$, where $u = [\omega]$ accordingly, only using the cohomology data of ω. We are going to denote $\mathcal{S}_u^{<0}$ by the subset of $\mathcal{S}_u$ having negative self-intersections.

Remark 2.2.
- *By Lemma 2.4 in [6], $\mathcal{S}_u = \mathcal{S}_w$.*
- *By the main theorem in [26], the negative self-intersection classes that admit embedded representatives are exactly those that have positive pairings with the class ω. Namely, we can find some integral class u' that admits some embedded curve in those classes of $\mathcal{S}_u^{<0}$, and the symplectic inflation of [26] allows us to change the class u' into u.*

Note that the way we do partition for rules surfaces is by looking at the inequality $u \cdot A \leq 0$, where $A \in \mathcal{S}_u^{<0}$ and $Cod(A) > 0$. In particular, by **wall**, we refer to those subsets $\{u | u \cdot A = 0, A \in \mathcal{S}_u^{<0}$ of the symplectic cone.

Here we draw the picture for the symplectic cone (with chambers) of a two-point blowup of $\Sigma_g \times S^2$:



Notice that in this figure, the $Y-$axis means the area ration μ, X and Z are blowup size of E_1, E_2 respectively. Clearly Y goes to ∞. The red triangles are the walls defined by the curve classes, and the tetrahedron are stable chambers in Conjecture 1.1.

3. Strata of $\mathcal{A}_\omega$ and stability under inflation

In this section, we prove the theorem 1.2 and 1.3 by inflating along embedded or nodal J-holomorphic curves.

3.1. **Curves.** Let us first recall several results from [35]. Here we translate Zhang's result into our basis $B, F, E_1, \cdots, E_n$ of $H_2(X, \mathbb{Z})$.

Theorem 3.1 (Theorem 1.1 or 3.4 in [35])**.** *Let M be an irrational ruled surface, and E an exceptional class. Then for any tamed J and any subvariety in class E, each irreducible component is a rational curve of negative self-intersection. Moreover, the moduli space $\mathcal{M}_E$ is a single point.*

Theorem 3.2 (Theorem 1.2 in [35])**.** *Let M be an irrational ruled surface of base genus $h \geq 1$. Then for any tamed J on M,*

(1) *there is a unique subvariety in the positive fiber class F passing through a given point;*

(2) *the moduli space $\mathcal{M}_F$ is homeomorphic to Σ_h, and there are finitely many reducible varieties;*

(3) *every irreducible rational curve is an irreducible component of a subvariety in class F.*

Notice that here a rational curve means the domain genus is 0, and by the adjunction formula each component has to be embedded.

Hence we have the following existence of curve result:

Proposition 3.3 (Proposition 3.6 in [35])**.** *There is a smooth section of the irrational ruled surface, i.e. there is an embedded J-holomorphic curve C of genus h such that $[C] \cdot F = 1$.*

This grants that there is an embedded curve in the class $B + kF - \sum_i c_i E_i$, where $k < g$.

Moreover, the discussion of [35] Corollary 3.3 confirms that $g_J(E) \geq \sum_i g_J(C_i)$, where g_J is the J-genus defined by $(e \cdot e + K_J \cdot e)/2 + 1$, and $E = \sum_i C_i$ is the cusp decomposition of E where C_i's are the irreducible components.

We summarize it as the following Lemma:

Lemma 3.4. *In ruled surfaces, the irreducible components of the stable curves of a Gromov limit for an exceptional curve all have $g_J = 0$.*

Proof. $0 = g_J(E) \geq \sum_i g_J(C_i)$, and each $g_J(C_i) \geq 0$. Hence all $g_J(C_i) = 0$. □

3.2. Stratification of $\mathcal{A}_\omega$.

The following statement is straightforward from the degeneration of rational curves:

For curves with prescribed singularities, after Definition 3.10 we are going to prove that the strata so that the fiber class curves with curve type E and $\{C, D\}$ are homological only. For the rest strata, we will show that they are subsets of Fréchet manifold with codimension 4 or higher.

Definition 3.5. *Let $\bar{e}$ be the collection of homology classes of the irreducible component of a stable curve in the class E, and let $\mathcal{A}_{\bar{e}}$ be the space of $J \in \mathcal{A}_\omega$ such that there is a stable (not necessarily embedded) curve of type $\bar{e}$ being J-holomorphic. Here, by a stable curve of type $\bar{e}$ and*

Lemma 3.6. *1) $\mathcal{A}_{\bar{e}}$ is an open subset of $\mathcal{A}_\omega$ if and only if $\bar{e} = \{E\}$.*
2) $\mathcal{A}_{\bar{e}}$ is of codimension 2 in $\mathcal{A}_\omega$ only if $\bar{e} = \{C, D\}$, where $C^2 = -2$ and $D \in \mathcal{E}$. Furthermore, in this case the representative in class C has to be an embedded (-2) sphere

Proof. Consider the stable curve in an exceptional class, $E = \sum_i C_i$ where each C_i is possibly multiple covered. Let $g_J(A)$ be the virtual genus of class A, given by $\frac{A \cdot A + K \cdot A}{2} + 1$. By Lemma 3.4, $0 = g_J(E) \geq \sum_i g_J(C_i) \geq 0$. Hence $g_J(C_i) = 0$ for each C_i. Then by the connectedness of $\sum_i C_i$, there must be at least one component with self-intersection at most -2. If this component is embedded and the only negative self-intersection -2 class, then it belongs to the Cod=2 part.

Otherwise, by the virtual dimension computation (Theorem 1.6.2 of [16] for example) and transversality for the underlying simple representative, the stratum of such J has codimension larger than 2. Here are more details:

If the only curve with square less than -1 is a simple class with self-intersection -2, then it has to be of $\{C, D\}$ by computing the square and pairing with K. More precisely, assume $E = C + \sum D_i + \sum P_j$, so that $C^2 = -2, D_i^2 \geq -1, P_j^2 \geq 0$ By $g_J(C) = g_J(D_i) = 0$, we have $K \cdot C = 0, K \cdot D_i = -1$ and $K \cdot P_J < -1$. Also, we have $K \cdot (C + \sum D_i + \sum P_j) = K \cdot E = -1$. Hence there can only be precisely one D_i.

For all other cases, let's first recall that for a simple class A, with a J-holomorphic representative, the index of A is given by $2g - 2 - 2K_J \cdot A$, where K_J

- If there are irreducible components with square less than -2.

Let $E = C_1 + \sum_{i>1} C_i$, such that $C_1^2 < -2$. If C_1 has a simple representative, then we are done. Now let's deal with the case C_1 is multiple covered. Let $C_1 = mC_1', m > 1$ such that C_1' has a simple representative. Notice that we immediately know that $(C_1')^2 \leq -1$.

Then we have $0 = 2g_J(C_1) = 2 + m^2(C_1')^2 + mK_J \cdot C_1'$. This means that

$$K_J \cdot C_1' = \frac{-2 - m^2(C_1')^2}{m} < 0.$$

Hence the simple representative has index less than 2. This means that $\mathcal{A}_{\bar{e}}$ has codimension greater than 2 in $\mathcal{A}_\omega$.

- If there are more than one components with square -2.

  Now let's assume that $E = C_1 + C_2 + \sum_{i>2} C_i$, such that $C_1^2 = C_2^2 = -2$. If both of them have simple representatives, then we are done. Now lets assume some of them are multiple covered, i.e. $C_1 = pC_1', C^2 = qC_2', p, q \geq 1$. Now we still have $0 = 2g_J(C_1) = 2 + p^2(C_1')^2 + pK_J \cdot C_1'$. This means that

$$K_J \cdot C_1' = \frac{-2 - q^2(C_1')^2}{q} < 0.$$

  Similarly, the index of C_2' is also non-positive. Hence both simple representatives has non positive indices. By the transversality of the simple representatives, $\mathcal{A}_{\bar{e}}$ has codimension greater than 2 in $\mathcal{A}_\omega$.

□

Now we are going to stratify our $\mathcal{A}_\omega$ as follows:

Definition 3.7. *We highlight some subsets of $\mathcal{A}_\omega$ and will prove that they behave well under Fredholm theory in Lemma 3.10.*

- *We call $\mathcal{A}_\omega^{top}$ to be the collection of $J \in \mathcal{A}_\omega$ characterized by the existence of an embedded J-holomorphic curve in $B + kF - \sum E_i$, $k \leq g$ and embedded curves in classes E_i and $F - E_i$.*
- *$\mathcal{A}_\omega^2$, is the collection of J characterized by 1) existence of exactly one exceptional class having stable representative of homology type $\{C, D\}$, s.t. $C^2 = -2, D^2 = -1$ as in Lemma 3.6, 2) all other exceptional classes embedded, and 3) $B + kF - \sum E_i$, $k \leq g$ embedded.*
- *$\mathcal{A}^{high} \subset \mathcal{A}_\omega$, and for any J here, there exists an embedded negative curve in the class $B - mF - \sum E_i$, so that $m > g$ or $B - mF - \sum E_i$ has a negative index, or there is a singular stable representative of the exceptional classes.*

Remark 3.8.
- *$\mathcal{A}_{high}$ is a collection of strata that are subsets of Fréchet submnaifolds with codimension 4 or higher.*
- *Cannot establish the full conjecture because $\mathcal{A}_{high}$ is not well understood in the general case.*
- *$\mathcal{A}_{high}$ is well understood in the $[\mu, 1, \frac{1}{2}, \cdots \frac{1}{2}]$ case.*

section class	embedded exceptional	not too badly deg	deg
$B+kF-\sum E_i, k\leq g$	$\mathcal{A}_\omega^{top}$	$\mathcal{A}_\omega^2$	$\mathcal{A}_\omega^{high}$
$B+mF-\sum E_i, m>g,$	$\mathcal{A}_\omega^{high}$	$\mathcal{A}_\omega^{high}$	$\mathcal{A}_\omega^{high}$

TABLE 1. Partition

Remark 3.9. *Note that the strata can be further decomposed but we won't do it.*

Lemma 3.10. *$\mathcal{A}_\omega$ is the disjoint union of the above 3 parts: $\mathcal{A}_\omega^{top}$ has Cod=0 in $\mathcal{A}_\omega$; the union of $\mathcal{A}_{S,\omega}$ has $Cod=2$ in $\mathcal{A}_\omega$, and the complement of $\mathcal{A}_\omega$ and the union of $\mathcal{A}_{S,\omega}$ has $Cod>2$.*

Proof. The $Cod>2$ part is an index computation. Since there is always an embedded curve in the class $B+kF-\sum E_i$, $k<g$. Then the index of such a curve is at least 4. This means that the corresponding strata have codimension at least 4.

The $Cod=2$ apparently has correct codimension in $\mathcal{A}_\omega$, by the index computation for an embedded (-2) sphere.

In particular, if J belongs to the Cod=2 part, by Lemma 3.6 there's exactly one exceptional curve E breaks into $C+D$ where C is the unique -2 curve and D is another exceptional curve.

Now we deal with the codimension 0 part, and show that there is a curve in class $B+kF-\sum E_i$, $k\geq g$.. Firstly, the existence of such a curve is guaranteed by Lemma 3.3 and the openness of the top strata comes from the fact that those higher codimension strata are closed.

□

Lemma 3.11. *For the exceptional class E, assume for a given J tames (or is compatible with) ω, it has homology type (2) in 3.6, i.e. the stable curve has two irreducible component classes C and D. They each has an embedded representative and intersects each other transversely. For such J, there is an J-tame (or compatible) inflation along the embedded curves in classes C, D such that ω' tames (is compatible with J) and $[\omega']=[\omega]+tP.D.(E), 0\leq t\leq \omega(E)$.*

Proof. See Appendix for a more general discussion. □

3.3. **Stability of Symp and inflation.** Firstly, note that one can always inflate along the curve in the class F, we have the following Lemma which allows us the find an inclusion between different $[\omega]$.

Lemma 3.12. *Let $u=[\mu,1,c_1,\cdots,c_k]$, and $u'=[\mu+\epsilon,1,c_1,\cdots,c_k], \epsilon>0$. Then $\mathcal{A}_u\subset\mathcal{A}_{u'}$.*

Direction	Strata	Class to inflate	Proof	Size/Note
$\uparrow$ or $\downarrow$	$\mathcal{A}_{u,\mathcal{C}}$	E_i, $F-E_i$,F	Lemma 3.15	Foliation and exceptional curve
$\uparrow$ or $\downarrow$	$\mathcal{A}_{u,open}$	$B+xF-\sum E_i$,$F-E_i$	Lemma 3.15	Foliation and exceptional curve
$\longrightarrow$	Any strata	F	Lemma 3.14	Foliation allows any size

TABLE 2. Inflation process

Proof. This is done by inflation along the embedded J-holomorphic curve in the class F, whose existence is granted by Lemma 3.2. □

Then we use the above partition of the space $\mathcal{A}_{[u]}$, when $\mu > g$ to obtain the following:

Proposition 3.13. *In the following cases, the strata have inclusion relations:*

(1) $\mathcal{A}_{u_1,\mathcal{C}} = \mathcal{A}_{u_2,\mathcal{C}}$, *if* $u_1 = [\mu, 1, c_i], u_2 = [\mu, 1, c_i]$, $\forall \mathcal{C} \subset S^{-2}$.

(2) $\mathcal{A}_{u,open} \supset \mathcal{A}_{u',open}$, *where* $u = [\mu, 1, c_i], u' = [\mu+\epsilon, 1, c_i]$, *and for all* $\mu > g, \epsilon > 0$.

(3) $\mathcal{A}_{u,\mathcal{C}} \supset \mathcal{A}_{u',\mathcal{C}}$, $u = [\mu, 1, c_i], u' = [\mu+\epsilon, 1, c_i]$, $\forall \emptyset \neq \mathcal{C} \subset S^{-2}$ *and for all* $\mu > 1, \epsilon > 0$.

Proof. (1) is covered Lemma 3.14. (2), (3)are covered by Lemma 3.15

□

Lemma 3.14. *For any stratum, including the open strata,* $\mathcal{A}_{u,\mathcal{C}} \subset \mathcal{A}_{u',\mathcal{C}}$, $u = [\mu, 1, c], u' = [\mu+\epsilon, 1, c]$, *and for all* $\mu > 1, \epsilon > 0$.

Proof. By [35] Theorem 1.6, we known that for each $J \in \mathcal{A}_{u,\mathcal{C}}$, through each point of M there is a stable J-holomorphic sphere representing the fiber class $F = [\mathrm{pt} \times S^2]$.

Then we can inflate along the embedded curve F. Let us start with $u = [\mu, 1, c]$.

Inflating, we obtain a form in $tP.D[F] + [\mu, 1, c]= [\mu+t, 1, c]$, for all $t \in [0, \infty)$.

□

Lemma 3.15. *For* $\mathcal{A}_{u,open} \supset \mathcal{A}_{u',open}$ *and* $\mathcal{A}_{u,C} \supset \mathcal{A}_{u',C}$, $C \in S^{-2}$, *where* $u = [\mu, 1, c_i], u' = [\mu+\epsilon, 1, c_i]$, *and for all* $\mu > g, \epsilon > 0$.

Proof. First, by Lemma 3.10, for any strata in this Lemma, there is an embedded curve in class $A = B + xF - \sum E_i$, where $x < g$.

Note that the process of inflating along this curve will increase the areas of the fiber class F and the E_j, if $E_j \cdot A > 0$. We can then inflate along E_j such that the area of E_j increases proportionally. Hence without loss of generality, we can assume $A = B + xF$, $x < g$. Also, note that x can be a negative number. Then we can inflate along it. And let's start with $u = [\mu, 1, c_i]$.

By inflating, we obtain a form in

$$tP.D[B+xF]+[\mu,x,c_i]+t_i[1,0,c_i]=[\mu+tx+\sum t_i,1+t,t_i+c_i],$$

which normalized to

$$\left(\frac{tx+\mu+\sum t_i}{1+t},1,\frac{c_i+t_i}{1+t}\right),$$

$\forall t\in[0,\infty)$.

Note that we will choose t_i/t proportional to c_i, that is, we always take $t_i=c_it$. Then in the resulting symplectic class, the area of the exceptional curves is always stable after normalization.

Then we just need to make sure that as long as $t\to\infty$, the resulting symplectic class covers $\mu\geq g$ cases. Note $\lim\limits_{t\to\infty}\dfrac{tx+\mu\sum t_i}{1+t}=x\leq g$. Hence we proved the statement of the Lemma. □

Proposition 3.16. *Let M be $\Sigma_g\times S^2\#k\overline{\mathbb{C}P^2}$ Suppose $\mu_i>g, i=1,2$ for ω_1, w_2, then the groups π_0 and π_1 of $Symp(M,\omega_1)$ and $Symp(M,\omega_2)$ are the same.*

Proof. Follows from Proposition 3.13 and the following commutative diagram:

$$\begin{array}{ccccc} Symp(M,\omega_1)\cap \mathrm{Diff}_0(M) & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_{\omega_1} \\ \downarrow & & =\downarrow & & \uparrow\downarrow \\ Symp_h(M,\omega_2)\cap \mathrm{Diff}_0(M) & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_{\omega_2} \end{array} \tag{2}$$

□

3.4. **Stability of equal size 1/2.**

We can prove the stronger version of the stability result for equal size 1/2.

Theorem 3.17. *Conjecture 1.1 holds for the form in class $[\mu,1,\frac{1}{2},\cdots,\frac{1}{2}],\mu>g$.*

Here we first describe the chamber structure and then provide a proof.

Notice that the space of such forms in class $[\mu,1,\frac{1}{2},\cdots,\frac{1}{2}],\mu>g$ is a line. The curves are given by $B-kF$ or $B-kF-\sum E_i$. Notice that each E_i has area $\frac{1}{2}$, and hence the chambers are those integer points or half integer points.

Hence the precise statement is the following:

Theorem 3.18. *The homotopy type of $G^g_{\mu,n}$ is constant for $\frac{k}{2}\leq\mu\leq\frac{k+1}{2}$, for any integer $k\geq 2g$. Moreover as μ passes the half integer $\frac{k+1}{2}$, all the groups $\pi_i, i=0,\cdots,2k+2g-1$ do not change.*

Proof. The proof is basically the same as that in Proposition 3.13. The only thing added here is the inflation on the higher codimensional strata, which is done by inflating along the embedded curve in class $B-kF-\sum E_i$.

The rest argument follows from Theorem 3.16. □

Explicit "fragile elements" in π_k could be detected by circle actions in [14] and Whitehead product. One can do this using a very similar approach as in [8].

4. Singular foliations and topological colimit for equal size blowups

The stability Theorem 3.16 grants us that the homotopy colimit $G^n_{\infty,g}$ (for each horizontal line fixing the blowup size) exists.

We are going to use the relationship between the space of singular foliations and the space of almost complex structures to establish a smooth diffeomorphism model for $G^n_{\infty,g}$. We will show that this smooth diffeomorphism model is disconnected and hence conclude that $G^n_{\infty,g}$ is disconnected.

4.1. The equal size $\frac{1}{2}$ blowup.

Proposition 3.14 and the following homotopy commutative diagram shows that the homotopy colimit exists:

$$(a)\quad \begin{array}{ccccc} G_u & \to & \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) & \to & \mathcal{A}_u \\ \downarrow & & \downarrow = & & \downarrow \\ G_{u'} & \to & \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) & \to & \mathcal{A}_{u'}, \end{array}$$

$$(b)\quad \begin{array}{ccc} G_u & \to & G_{u'} \\ & \searrow & \downarrow \\ & & G_{u''} \end{array}$$

Remark 4.1. *Zhang's Lemma in [35] provides that for each J there is a J-holomorphic singular foliation when we have an equal size blowup of $\frac{1}{2}$.*

This is because any exceptional curve has the minimal area in this case and they can never degenerate.

We are going to use the following singular foliation in the smooth (topological) category to prove that the colimit is not connected:

Definition 4.2. *A* **singular foliation** *by S^2 of $\Sigma_g \times S^2\#\overline{\mathbb{C}P^2}$ is defined as a foliation with smooth embedded spherical leaves in the $F = [pt \times S^2]$ class and one nodal leaf with two embedded spherical components, each in the class E and $F - E$ respectively. Also, we require that the complement of the singular leaf is a smooth foliation over Y which is a compact curve of genus g except on a single point.*

Let $\mathcal{F}_{std}$ be the standard blow up foliation by J_{std}-holomorphic leaves. Note that if we blowdown the complex structure, we obtain the split complex structure on $\Sigma_g \times S^2$, and the induced foliation is the split foliation by the spheres.

Following verbatim the argument in [9], we have the following Lemma on the space of foliations and transitive action, when there is only finitely many nodal fibers.

Lemma 4.3. *Let Fol_0 be the connected component of Fol that contains $\mathcal{F}_{std}$. $\mathcal{A}_\infty$ is weakly homotopic to Fol_0.*

Proof. Observe that there is a map $\mathcal{A}_\infty \to \mathrm{Fol}_0$ given by taking J to the singular foliation of $M_g \# n\overline{\mathbb{C}P^2}$ by J-spheres in class F or $F-E$. Standard arguments in [32] Ch 2.5 show that this map is a fibration with contractible fibers. Hence it is a homotopy equivalence. □

Lemma 4.4. *There is a transitive action of* $\mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$ *on* Fol_0.

Proof. Since $S^2 \setminus pt$ is compact and simply connected, each generic leaf of this foliation has trivial holonomy and hence has a neighborhood that is diffeomorphic to the product $D^2 \times S^2$ is equipped with the trivial foliation with leaves $pt \times S^2$.

Since our foliation has smoothly embedded leaves and only one nodal leaf, we can find a 2-form transverse to each leaf. And the Poincaré dual of such 2-form is a smooth section, not passing through the singular point p.

Now let's take an arbitrary singular foliation $\mathcal{F}' \in \mathrm{Fol}_0$ and denote the smooth section by Σ'. We'll prove that $\mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$ takes this foliation $(\mathcal{F}', \Sigma')$ to $\mathcal{F}_{std}, \Sigma_{std}$ where Σ_{std} is the smooth section (which is indeed J_{std}-holomorphic).

Since $\mathcal{F}'$ and $\mathcal{F}_{std}$ are in the same path connected component, there is a $\phi \in \mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$ sending Σ' to Σ_{std}, such that the singular leaf of $\mathcal{F}'$ goes to the singular leaf of $\mathcal{F}_{std}$ while the two singular points are identified. Now let's fix a finite covering $\{D_i, 1 \le i \le n\}$ of Σ', such that the local foliations over D_i's cover the manifold $\Sigma_g \times S^2 \# \overline{\mathbb{C}P^2}$.

Then we use partition of unity for the covering $\{D_i, 1 \le i \le n\}$ of Σ', and for each local foliation, we apply a ϕ_i such that the foliation $\mathcal{F}'$ under $\phi \circ \phi_1 \circ \cdots \circ \phi_n$ agrees with the foliation $\mathcal{F}_{std}$.

Now we have the transitive action of $\mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$ on Fol_0. Notice that this action of $\mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$ does not necessarily preserve the leaf. □

Hence there is a fibration sequence

$$\mathcal{D} \cap \mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2}) \to \mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2}) \to \mathrm{Fol}_0, \tag{3}$$

where $\mathcal{D}$ is the diffeomorphism preserving the leaves in the foliation $\mathcal{F}_{std}$. We denote this fiber group by $\mathcal{D}^n_g$.

Definition 4.5. $\mathcal{D}^n_g$ *is the elements in the identity component of the diffeomorphisms which fit into the commutative diagram*

$$\begin{array}{ccc}
M_g\#n\overline{\mathbb{C}P^2} & \xrightarrow{\phi} & M_g\#n\overline{\mathbb{C}P^2} \\
\downarrow & & \downarrow \\
(M_g, \{p_1, \cdots, p_n\}, F_p) & \xrightarrow{\phi'} & (M_g, \{p_1, \cdots, p_n\}, F_p) \\
\downarrow & & \downarrow \\
(\Sigma_g, pt) & \xrightarrow{\phi''} & (\Sigma_g, pt).
\end{array}$$

Here p_i is the intersection point $E_i \cap (F - E_i)$ of the singular fiber. And the first level of the downward arrow means that we contract the E_i component. We abuse notation here to still denote p_i for the point in M_g after contracting the curve E_i.

On the second level, ϕ' is a diffeomorphism of M_g keeping the points p_i fixed and fixing the fiber F_p passing through p_i fixed as a set, and preserves other leaves in the standard foliation.

The base Σ_g is the holomorphic curve B_{std} w.r.t the standard complex structure, and the map downward is obtained by firstly blow down the exceptional sphere and then projects down to the base curve.

Proposition 4.6. *Take $M_g\#n\overline{\mathbb{C}P^2}$ with a form in the class $[\mu, 1, \frac{1}{2}, \cdots, \frac{1}{2}]$. Then let μ go to ∞.*

(1) *$\mathcal{D}_g^n$ is weakly homotopic to $G_{\infty,g}^n$.*
(2) *The group $\mathcal{D}_g^n$ is disconnected when $g \geq 2$.*
(3) *When $\mu \to \infty$, s.t. for $i = 0, 1$, $\pi_i(G_{u,g}^n) = \pi_i(G_{\infty,g}^n)$ for $i \leq min\{Cod(u)\} - 1$, and hence the groups $G_{u,g}^n$ are disconnected for $g \geq 2$.*

Proof. For statement (1), note the equation (3) fits into the commutative diagram:

$$\begin{array}{ccc} \mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2}) & \to & \mathcal{A}_\infty \\ \downarrow & & \downarrow \\ \mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2}) & \to & \mathrm{Fol}_0, \end{array}$$

where the upper map is given as before by the action $\phi \mapsto \phi_*(J_{\mathrm{std}})$. Hence there is an induced homotopy equivalence from the homotopy fiber $G_{\infty,g}^1$ of the top row to the fiber $\mathcal{D}_g^1$ of the second.

To prove statement (2), first note that we have the following fibration

$$\mathrm{Diff}(\Sigma_g, p_1, \cdots, p_n) \longrightarrow \mathrm{Diff}(\Sigma_g) \longrightarrow Conf(\Sigma_g, n),$$

Where $Conf(\Sigma_g, n)$ is the configuration of n points on Σ_g.

Taking the right portion of the Long Exact sequence, we have:

$$1 \longrightarrow \pi_1(Conf(\Sigma_g, n)) \longrightarrow \pi_0[\mathrm{Diff}(\Sigma_g, p_1, \cdots, p_n)] \longrightarrow \pi_0[\mathrm{Diff}(\Sigma_g)] \longrightarrow 1.$$

Then restricting to $\mathrm{Diff}_0(\Sigma_g)$, we obtain an element in the identity component of $\pi_0(\mathrm{Diff}(\Sigma_g))$ but not in the identity component of $\pi_0(\mathrm{Diff}(\Sigma_g, p_1, \cdots, p_n))$, where $p_1, \cdot, p_n$ are the points we will blow-up.

It can be explicitly in the following way: choose a path $\alpha(t) \subset \mathrm{Diff}(\Sigma_g), t \in [0, 2\pi]$, pushing $p_1, \cdots, p_n$ along homologically non-trivial loop in $Conf(\Sigma_g, n)$. Now $\alpha(0) = id$ and $\alpha(2\pi) \in \mathrm{Diff}(\Sigma_g, p_1, \cdots, p_n) \cap \mathrm{Diff}_0(\Sigma_g)$ and note that $\alpha(2\pi)$ is the desired element.

Next, we lift the path $\alpha(t)$ into dimension 4. To do that, first fix M_g, Σ_g and choose J_{split}. There is a natural family $\alpha(t) \times id \subset \mathrm{Diff}_0(M_g)$, which act on the leaves in the trivial manner. For each of t, we have a product complex structure on M_g by pulling back J_{split} by $\alpha(t) \times id$. We are going to obtain a family of complex structures by blowing up at the points $\alpha(t)|_{p_0} \in M_g$. This gives us a loop of complex structures J_t on $M_g\#\overline{\mathbb{C}P^2}$ where $J_0 = J_{\mathrm{std}}$. Note that by [35], each J_t gives rise to a singular foliation $\mathcal{F}_t$, as in Definition

4.2. Geometrically, $\mathcal{F}_t$ is a loop in Fol_0 starting with the standard singular foliation $\mathcal{F}_{std}$, pushing the marked point p along a homological non-trivial circle on the standard base Σ_g for time $t \in [0, 2\pi]$.

By the transitivity Lemma 4.4, we can use a path ϕ_t in $\mathrm{Diff}_0(M_g \# n\overline{\mathbb{C}P^2})$ to push $\mathcal{F}_0$, so that $\phi_t \circ \mathcal{F}_0 = \mathcal{F}_t$. Note that ϕ_t in $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2})$ pushes the standard foliation along this loop.

Now we focus on the diffeomorphism $\phi_{2\pi}$. First note that $\phi_{2\pi}$ preserves the singular foliation $\mathcal{F}_{std}$, since the foliation $\mathcal{F}_{2\pi} = \mathcal{F}_0 = \mathcal{F}_{std}$. Hence $\phi_{2\pi} \in \mathcal{D}^1_g$. Also, the above paragraph gives an explicit isotopy of $\phi_{2\pi}$ to the identity map in $\mathrm{Diff}_0(M_g \# n\overline{\mathbb{C}P^2})$, through the path ϕ_t.

We now show that $\phi_{2\pi}$ is not isotopic to id in $\mathcal{D}^1_g$. Suppose there is an isotopy to id, then by path lifting of the fibration 3, we would have a leaf-preserving element in $\mathrm{Diff}_0(M_g \# n\overline{\mathbb{C}P^2})$, so that it is isotopic to identity through a path in $\mathcal{D}^n_g$. Furthermore, this path pushes the given foliation along the lifting of the loop $\mathcal{F}_t, t \in [0, 2\pi]$. Now apply the diagram of definition 4.5. We would have an isotopy that would in turn give an isotopy of (Σ_g, p), connecting the time 2π diffeomorphism to identity. This is a contradiction against the Birman exact sequence. Hence statement (2) holds.

Statement (3) follows from the stability Theorem 3.16. □

Remark 4.7. *When $g = 0$, one can blow up $S^2 \times S^2$ at k points with equal sizes. It is shown in [22] that when $k \leq 3$, $G^k_{u,0}$ is connected for all ω. When $k > 3$, $\pi_0 Symp_h$ (for a type $\mathbb{D}$ form, which amounts to blowup equal and 1/2 of the size of the fiber) is a braid group of k strands on spheres (cf. [20]). This follows the same pattern as $\mathrm{Diff}(S^2, k)$, which is the diffeomorphism group of S^2 fixing k points. The techniques there are ball swappings. As pointed out in Example 2.3 of [23], there is a way to construct ball swappings of a ball along a non-trivial loop in Σ_g. It is an interesting question to explore whether the construction here is indeed a ball swapping map. And it will be more exciting to prove using either construction that the $\pi_0 \mathcal{D}^n_g$ is a braid group of n strands on Σ_a.*

Appendix A. Constructing infinite dimensional chart for subsets in $\mathcal{A}_\omega$

Here we also provide a proof, which addresses the Fréchet (Banach) local chart.

Claim A.1. *In Lemma 3.6, if $\bar{e}$ is not $\{E\}$ or $\{C, D\}$, then $\mathcal{A}^l_{\bar{e}}$ is a analytic subset of a submanifold which has codimension larger than 2 in $\mathcal{A}^l_\omega$. Here $\mathcal{A}^l_\omega$ and $\mathcal{A}^l_{\bar{e}}$ are the C^l counterpart of $\mathcal{A}_\omega$ and $\mathcal{A}_{\bar{e}}$ respectively.*

Proof. With the C^∞ topology, $\mathcal{A}_\omega$ is a paracompact infinite-dimensional Fréchet manifold (smooth manifold locally modeled by Fréchet spaces).

We'll follow the notation and idea of Taubes [34] Lemma A.1-3, and also the idea of [31] chapter 3. Let's denote the space $(\bar{e}, M)$ as smooth maps representing spherical classes $\{[C_1], [C_2], \cdots, [C_n]\} = \bar{e}$ with prescribed singularities. $(\bar{e}, M)_*$ by the space of somewhere injective maps representing the same classes that belong to $W^{1,p}, p > 2$ category. Denote $\mathcal{P}$ as the universal moduli space of simple smooth pseudoholomophic maps representing classes

$\{[C_1], [C_2], \cdots, [C_N]\} = \bar{e}$, and $\mathcal{J}^l$, $\mathcal{P}^l$ by the $W^{l,p}$ completion of $\mathcal{J}_\omega$ and $\mathcal{P}$ respectively. Following Taubes [34] Lemma A.2 and A.3, the Banach space topology of $\mathcal{P}^l$ coming from inclusion in $C^{l-2}(\bar{e}, M) \times \mathcal{J}^l$ is equivalent to the topology coming from the inclusion in $(\bar{e}, M)_* \times \mathcal{J}^l$. Meanwhile, the Fréchet topology of $\mathcal{P}$ coming from C^∞ topology on $(\bar{e}, M)$ and $\mathcal{J}_\omega$ is equivalent to the topology coming from the inclusion in $(\bar{e}, M)_* \times \mathcal{J}_\omega$.

So now we can take everything as subspaces of $(\bar{e}, M)_* \times \mathcal{J}^l$, where $1 < l \leq \infty$. For any given l, we know $\mathcal{P} \subset \mathcal{P}^l \subset (\bar{e}, M)_* \times \mathcal{J}^l$, and take the $\bar{\partial}$ operators $\mathcal{F}(u_i, J) = \bar{\partial}_J u_i, 1 \leq i \leq N$. Consider the differentials

$$D\mathcal{F}(u_i, J) : W^{k,p}(S^2, u_i^* TM) \times C^l(M, End(TM, J, u_i)) \to W^{k-1}(S^2, \Lambda^{0,1} \otimes_J u_i^* TM).$$

Note that on $\mathcal{P}^l$, $\mathcal{F}(u_i, J)'s$ simultaneously vanish, for any $1 \leq k \leq 1-1$ by elliptic regularity. The main result proved in [15] implies that each $D\mathcal{F}(u_i, J)$ is injective since every class $[C_i]$ has negative square. Since each $D\mathcal{F}(u_i, J)$ is a Fredholm operator with constant rank, by implicit function theorem ([18] Chapter 2) in the Banach setting, there exist local inverse of $\bar{\partial}$ and this endows $\mathcal{P}^l$ a Banach chart. As l getting larger, this will endow $\mathcal{P}$ a sequence of Banach charts. And when $l = \infty$, on each local chart, the inverse limit of the Banach spaces will become a Fréchet space and they patch together to endow $\mathcal{P}$ a Fréchet manifold structure.

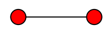
Then the rest proof will follow from the index computation of projection $\mathcal{P}^l \to \mathcal{J}^l$ in the Banach setting and then taking the inverse limit. The argument is written down in [31] p151 Proof of Theorem 6.2.6(II) and [5] Appendix B.1. We point out here the projection $\pi : \mathcal{P} \to \mathcal{J}_\omega$ is an embedded submanifold. First, it is injective by [15], then it is an immersion by the construction of the local chart. Then we restrict π to its image and consider the inverse $\pi^{-1}_{im\pi}$. Suppose there is a sequence $\{J_n\} \subset im\pi$ converging to $J_0 \in im\pi$, then by Gromov compactness their preimages $\{C_n\} \subset \mathcal{P}$ of π also converge to C_0. Note C_0 is J_0 holomorphic, which means each class $[C_i]$ has an embedded J_0 holomorphic representative. Since each $[C_i] \in S^{\leq -1}_\omega$, C_0 must belong to $\mathcal{P}$. This means $\pi^{-1}_{im\pi}$ preserves limit and hence it is a continuous map. Then $\pi : \mathcal{P} \to \mathcal{J}_\omega$ is homeomorphic onto its image, and hence an embedding. Also, from the way that the Fréchet manifold structure is given, it is paracompact (by the Morita theorem, Lindelöf manifolds modeled by smoothly regular spaces are smoothly paracompact). □

Remark A.2.
- *As denoted by [2], since all the Fréchet manifolds we work with can naturally be interpreted as inverse limits of Banach manifolds; and the successive inclusions between the Banach manifolds are weak equivalences. Then the results about $\mathcal{A}_{\bar{e}}$'s stated in the smooth setting can be interpreted as the corresponding result for each Banach manifold indexed by each k, and then apply the weak equivalences of the algebraic invariants between the Fréchet object and the sequence of Banach objects.*
- *One can also find versions of inverse function theorems in the tame Fréchet category, for example, Richard Hamilton [13] in the general cases, and Gerstenberger [11] section 5 deals with Cauchy-Riemann operators between tame Fréchet spaces in our current setting.*

Appendix B. Inflation theorems along singular curves

Theorem B.1. *For a symplectic four manifold (M^4, J, τ_0) such that J is a τ_0-compatible almost complex structure. Assume that M admits embedded J-holomorphic curves $u_i : (\Sigma_i, j_i) \to (M^4, J), i = 1, 2$ whose images are Z_1, Z_2 respectively. Let's also denote their homology classes Z_1 and Z_2 with $Z_i^2 = -m_i$ and $Z_1 \cdot Z_2 = 1$. For all ϵ_1 and ϵ_2 there exist a family of symplectic forms $\tau_{\mu,\eta}$ all taming J and $[\tau_{\mu,\eta}] = [\tau_0] + \mu P.D.[Z_1] + \eta P.D.[Z_2]$ for all $0 \le \mu \frac{\tau_0(Z_1)}{m_1} - \epsilon_1$ and $0 \le \eta \frac{\tau_0(Z_2)}{m_2} - \epsilon_2$. Here $P.D.[Z_i]$ is the Poincaré dual of Z_i.*

Proof. Now let's focus on the case when D only has two components intersecting transversely, i.e. the augmented graph being



Take $N(Z_i)$ to be a neighborhood of Z_i consisting of the unit disk bundle over the curve in class Z_i. Let's call the unit disk bundle $U(Z_i)$. Denote by r_i the radial coordinate of $U(Z_i)$. We assume $\tau_0(Z_1) = 1$ and $\tau_0(Z_2) = b$. Denote by σ_{Z_1} and σ_{Z_2} the area form on Z_1 and Z_2 such that $\int_{Z_1} \sigma_{Z_1} = 1$ and $\int_{Z_b} \sigma_{Z_2} = b$. We then can choose connections on the disk bundles such that the connection one-forms α and β on the bundles over Z_1 and Z_2 obey $d\alpha = m_1 \pi^*(\sigma_{Z_1})$ and $d\beta = m_2 \pi^*(\sigma_{Z_2})$ where π being the bundle projections (there's no confusion so we do not distinguish) respectively.

Now in a very small tubular neighborhood of $Z_1 \cup Z_2$, by the main theorem of [33], we can choose the symplectic form τ_0 to be diffeomorphic via ϕ to the following:

- near Z_1 and away from Z_2, $\tau_0 \sim (1 + m_1 r_1^2)\pi^*(\sigma_{Z_1}) + 2r_1 dr_1 \wedge \alpha$;
- near Z_2 and away from Z_1, $\tau_0 \sim (1 + m_2 r_2^2)\pi^*(\sigma_{Z_2}) + 2r_2 dr_2 \wedge \beta$;
- in the intersection neighborhood (product of two disks), $\tau_0 \sim 2r_1 dr_1 \wedge \alpha + 2r_2 dr_2 \wedge \beta$.

Note that by doing the diffeomorphism ϕ we only changed the coordinates on $N(Z_1)$ and $N(Z_2)$, we only changed the parametrization, but not the form.

Note that τ_0 on the product of two disks perfectly matches the forms on the Z_1 neighborhood and the Z_2 neighborhood, when restricted to Z_1 and Z_2.

Then for the inflation form $\tau_{\mu\eta}$, we will do the similar modification as [7], using two functions with the following properties(proved in [7] section 2):

- The functions $f_\mu(r)$ $g_\tau(r)$ will be nonincreasing positive functions of r supported in a neighborhood $r \le r_0$, constant in a smaller neighborhood near $r = 0$
- $f_\mu(r) = M < \frac{1}{m} - \epsilon$.
- $f_\mu(r)$ has a uniform bounded of its derivative at every order in the interval $[0, r_0]$.

We are going to change the forms in the following way:

- near Z_1 and away from Z_2, $\tau_{\mu\eta} = (1 + m_1 r_1^2 - m_1 f(\mu))\pi^*(\sigma_{Z_1}) + [2r_1 - f'_\mu(r_1)]dr_1 \wedge \alpha$;
- near Z_2 and away from Z_1, $\tau_{\mu\eta} = (1 + m_2 r_2^2 - m_2 g(\eta))\pi^*(\sigma_{Z_2}) + [2r_2 - g'_\eta(r_2)]dr_2 \wedge \beta$;

Those are the same treatment and the same quadratic estimate will work.

Then, in the red intersection neighborhood (product of two disks), $\tau_{\mu\eta} = [2r_1 - f'_\mu(r_1)]r_1 dr_1 \wedge \alpha + [2r_2 - g'_\eta(r_2)]r_2 dr_2 \wedge \beta$.

The argument basically follows from [30] Lemma 5.2.1. The first bullet grants that the cohomology class, and the 3rd bullet grants there is a Moser interpolation between the blue part and the red part.

We further check the positivity.

Lemma B.2. *$[2r_1 - f'_\mu(r_1)]$ and $[2r_2 - g'_\eta(r_2)]$ are non-decreasing functions in r_1, r_2, and they have bounded derivatives at every order.*

Proof. Follows from properties of $f_\mu(r)$ and $g_\eta(r)$. □

Note that we want
1) the form $\tau_{\mu\eta}$ obtained both ways on the boundary disks match each other.
2) on the Z_1 and Z_2, $\tau_{\mu\eta}$ scale τ_0 as desired.
3) it has the correct cohomology class.
And it is easy to check the above form satisfies both conditions.

Now we are going to prove the tameness using the quadratic estimate in the neighborhood of the product of disks $S = D_1 \times D_2$, where $0 \in D_1 \subset Z_1, \quad 0 \in D_2 \subset Z_2$: We'll use the splitting of the tangent space $T_p(S) = E_1 \oplus E_2$, where E_i tangents to D_i.

Under this choice of splitting, let's assume that

$$J_p = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are 2×2 matrices.

Now let's do the general computation and let

$$\mathcal{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\begin{aligned} &\tau_{\mu\eta}((v,w), J_p(v,w)) \\ &= F(\mu,\eta,r_1)2r_1 dr_1 \wedge \alpha((v,w), J_p(v,w)) + G(\mu,\eta,r_2)2r_2 dr_2 \wedge \beta((v,w), J_p(v,w)) \end{aligned}$$

$$= Fv^\top \mathcal{A}^\top Av + Fv^\top \mathcal{A}^\top Bw + Gw^\top \mathcal{A}^\top Cv + Gw^\top \mathcal{A}^\top Dw. \tag{4}$$

We want to prove this is positive at least for the neighborhood where both disks have sufficient small radius.

Note that since the curve at $r_1 = 0$ or $r_2 = 0$ is J-holomorphic we can assume that the fiber disks are locally J-holomorphic near $r_1 = 0$ or $r_2 = 0$ and this means $B = C = 0$ when $r_1 = 0$ or $r_2 = 0$.

Also note that by the standardization of the neighborhood, we achieved both ω orthogonal and J-orthogonal. Since we started with triple (N, ω, J) where N is the neighborhood with two J-holomorphic curves intersecting at one point. Then we use a diffeomorphism supported along N, making two curves ω' orthogonal. This diffeomorphism also pushes forward J, and both curves are still J'-holomorphic. Since near the intersection point, the disk on one curve is the base and the disk on the other curve is the fiber; we also know that they are both J'-holomorphic after the push forward. In the above (and below) argument, we still use J to denote J'.

To justify $B = C = 0$, we only need to show that J preserves the base and fiber. And this is the J-holomorphic condition. And this means that the local J matrix has to be blockwise diagonal.

Then we know that

$$\tau_0((v,w), J_p(v,w)) = v^\top \mathcal{A}^\top Av + w^\top \mathcal{A}^\top Dw > 0$$

Since

$$J_p = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^2 = -Id,$$

when $r_1 = 0$ or $r_2 = 0$, we can find a neighborhood $r_1, r_2 < \delta$ and a positive constants K and L depending only on J s.t.

$$||v||^2 \leq Kv^\top \mathcal{A}^\top Av, \quad ||w||^2 \leq Kv^\top \mathcal{A}^\top Dv,$$

and

$$v^\top \mathcal{A}^\top Bw \leq L||v||||w||, \quad w^\top \mathcal{A}^\top Cv \leq L||v||||w||.$$

Then the above shows that for sufficient small neighborhood, i.e. $r_1, r_2 < \delta$, equation (4) can be made positive, because F, G as functions are uniformly bounded with value greater than 1.

□

Remark B.3. *A result of Guadagni in [12] provides a neighborhood theorem for singular curves (which allows cycles beyond chain types) to improve the singular inflation theorem.*

Since the homological intersections are one, by positive intersection, we have a divisor $D \subset M$, which is normal crossing, and has no cycle in its augmented graph (Γ, a) (each component being a node and each intersection being an edge).

Now, if we have an ω'-orthogonal divisor (D', ω') with augmented graph $(\Gamma, \vec{1})$, which is the same as that of the neighborhood triple (X, ω, D), then there exist neighborhood N' of D' symplectomorphic to a neighborhood of D and sending D' to D (See [33] and [10]).

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