STABILITY OF THE SYMPLECTOMORPHISM GROUP OF RATIONAL SURFACES

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ABSTRACT. We apply Zhang's almost Kähler Nakai-Moishezon theorem and Li-Zhang's comparison of J-symplectic cones to establish a stability result for the symplectomorphism group of a rational surface with Euler number up to 12.

CONTENTS

1. INTRODUCTION

Let M be a closed, oriented, smooth 4-manifold and ω a symplectic form on M. Then the group of orientation-preserving diffeomorphisms $\mathrm{Diff}^+(M)$ with the standard C^∞-topology is an infinite-dimensional Fréchet Lie group. And the subgroup of symplectomorphisms, denoted by $Symp(M,\omega)$, is also an infinite-dimensional Fréchet Lie group.

Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic spheres and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,

$$K_\omega \cdot A = -A \cdot A - 2. \tag{1}$$

We introduce the following subsets of S_ω as in [16]: let

$$\mathcal{S}_\omega^{\geq n}, \quad \mathcal{S}_\omega^{>n}, \quad \mathcal{S}_\omega^{n}, \quad \mathcal{S}_\omega^{\leq n}, \quad \mathcal{S}_\omega^{<n}$$

be the subsets of $\omega-$symplectic spherical classes with square $\geq n, > n, = n, \leq n, < n$ respectively. The set S_ω^{-2} turns out to be very useful in the study of π_0 and π_1 of $Symp(M,\omega)$ where M is a rational

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4-manifold with $\chi(M) \leq 8$, ([16] [18]). In this paper we continue to investigate how the symplectic spherical set $S_\omega^{\leq -n}, n > 2$, is related to higher π_i's.

Here is the main result of this note:

Theorem 1.1. *Let M be a rational surface with $\chi \leq 12$ and ω and ω' two symplectic forms on M. Suppose $\mathcal{S}_\omega^{\geq -n} = \mathcal{S}_{\omega'}^{\geq -n}, n \geq 2$. Then the groups π_i of $Symp(M,\omega)$ and $Symp(M,\omega')$ are the same for $0 \leq i \leq 2n-3$.*

In particular, if $\mathcal{S}_\omega = \mathcal{S}_{\omega'}$, then $\pi_i(Symp(M,\omega)) = \pi_i(Symp(M,\omega'))$ for all $i \geq 0$.

This result generalizes the stability results in [2, 14, 4, 5, 3, 16, 18], etc. In particular, it implies that, when $\chi \leq 12$, both π_0 and π_1 of $Symp(M,\omega)$ depend only on $S_\omega^{\geq -2}$. Although the stability criterion alone does not compute $\pi_i(Symp(M,\omega))$, it does offer significant insight to understanding the rational homotopy type of $Symp(M,\omega)$, which is already rather complicated starting when $\chi \geq 4$ (see [14, 5, 3] for $\chi = 5, 6, 7$ respectively).

Our proof uses a variation of the inflation strategy in [24] that is tailored to the rational or ruled surfaces, and the cone Theorem in [30] greatly simplified the inflation process. Also, the comparison of J-symplectic cones in [23] plays an important role.

One new feature is that we systematically explore various spaces/groups associated to a symplectic class u, rather than a symplectic form ω. This is based on the fact that cohomologous symplectic forms on a rational or ruled surface are diffeomorphic (cf. [13, 21], and the surveys [20, 27]). It follows that the corresponding symplectomorphism groups are conjugate subgroups of $\mathrm{Diff}^+(M)$ and hence isomorphic as groups and homeomorphic as topological spaces.

We now introduce a few spaces/groups associated to a symplectic class u and reformulate Theorem 1.1 accordingly. Let M be a closed, oriented, smooth 4–manifold and Ω_M the space of orientation-compatible symplectic forms. The symplectic cone $\mathcal{C}_M \subset H^2(M;\mathbb{R})$ is the set of classes of orientation-compatible symplectic forms. Clearly, it is contained in the positive cone $\mathcal{P}_M = \{e \in H^2(M;\mathbb{R}) | e\cdot e > 0\}$.

Definition 1.2. *For $u \in \mathcal{C}_M$, let Ω_u denote the space of symplectic forms in the class u.*

- *Let $\mathcal{S}_u = \cup_{\omega\in\Omega_u}\mathcal{S}_\omega$, and define the subsets $\mathcal{S}_u^{\geq n}$, $\mathcal{S}_u^{>n}$, $\mathcal{S}_u^{n}$, $\mathcal{S}_u^{\leq n}$, $\mathcal{S}_u^{<n}$ similarly.*
- *For a rational or ruled surface M, let $Symp(M,u) = Symp(M,\omega)$ for $\omega \in \Omega_u$. As remarked above, $Symp(M,u)$ is well defined up to isomorphism algebraically and homeomorphism topologically.*

For a rational or ruled surface, we will show in Proposition 2.4 that $\mathcal{S}_u = \mathcal{S}_\omega$ for any $\omega \in \Omega_u$. With this understood, Theorem 1.1 can be restated as

Theorem 1.3. *Let M be a rational surface with $\chi \leq 12$ and $u, u' \in \mathcal{C}_M$. Suppose $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}, n \geq 2$. Then the groups π_i of $Symp(M,u)$ and $Symp(M,u')$ are the same for $0 \leq i \leq 2n-3$.*

In particular, if $\mathcal{S}_u = \mathcal{S}_{u'}$, then $\pi_i(Symp(M,u)) = \pi_i(Symp(M,u'))$ for all $i \geq 0$.

We will actually prove this version. For this purpose, we also introduce

Definition 1.4. *Fix $u \in \mathcal{C}_M$.*

- *Let $\mathrm{Diff}_u(M)$ denote the group of the diffeomorphisms fixing the class u.*
- *Let $\mathcal{A}_u$ be the space of almost complex structures that are compatible with some $\omega \in \Omega_u$.*

$\mathrm{Diff}_u(M)$ is contained in $\mathrm{Diff}^+(M)$ since $u \in \mathcal{C}_M \subset \mathcal{P}_M$. Clearly, $\mathrm{Diff}_u(M)$ acts on Ω_u. Moreover, for a rational or ruled surface, the action is transitive since cohomologous symplectic forms are diffeomorphic (necessarily by a diffeomorphism preserving the cohomology class). Since $Symp(M,\omega) \subset \mathrm{Diff}_u(M)$, the isotropy group at $\omega \in \Omega_u$ is just $Symp(M,\omega)$. Therefore we have the following fibration for a rational or ruled surface M, a class $u \in \mathcal{C}_M$ and a symplectic form $\omega \in \Omega_u$:

$$Symp(M,\omega) \to \mathrm{Diff}_u(M) \to \Omega_u. \tag{2}$$

We will relate the terms in the fibration (2) with $\mathcal{S}_u$ and $\mathcal{A}_u$ in Sections 2 and 4. This fibration could be compared to Kronheimer's fibration (21) in [12], which is valid for all closed symplectic manifolds and has been used in previous stability results. We will outline an approach to Theorem

1.1 via the fibration (21) in Section 5. The fibration (2), valid only for a rational or ruled surface, seems to provide a more natural approach to Theorem 1.1.

For the cases $\chi(M) > 12$, see Remark 5.5, Conjecture 5.10 and the comments there.

Convention. Throughout the paper, M is a closed, oriented, smooth 4-manifold and ω is an orientation-compatible symplectic form. We will often identify an integral degree 2 homology class with an integral degree 2 cohomology class via Poincaré duality. And we use the dot product to denote various pairings.

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2. $\mathcal{S}_u^{\geq -2}$ DETERMINES $\mathrm{Diff}_u(M)$

In this section, for $u \in \mathcal{C}_M$, we introduce the canonical class K_u, and study the group $\mathrm{Diff}_u(M)$ and the set $\mathcal{S}_u$ for a rational or ruled surface M. In particular, we show that the groups $\mathrm{Diff}_u(M)$ and $Symp(M,\omega)$ have the same homological action (Lemma 2.2) and the subset $\mathcal{S}_u^{\geq -2}$ determines both $\mathrm{Diff}_u(M)$ and K_u (Proposition 2.5).

2.1. K_u and $\mathrm{Diff}_u(M)$.

Lemma 2.1. *Suppose M is a closed, smooth 4-manifold and $u \in \mathcal{C}_M$. Then symplectic forms in Ω_u have the same symplectic canonical class, which we denote by K_u. Moreover, $\mathit{Diff}_u(M)$ preserves K_u.*

Proof. The first statement follows directly from [29] when $b^+(M) > 1$ and Proposition 4.1 in [21] when $b^+(M) = 1$, which says that $\mathcal{C}_{M,K} \cap \mathcal{C}_{M,K'} = \varnothing$ if $K \neq K'$. Here, for $K \in H^2(M;\mathbb{Z})$, the K−symplectic cone $\mathcal{C}_{M,K} = \{e \in \mathcal{C}_M | e = [\omega] \text{ with } K_\omega = K\}$.

The second statement follows from the first since $K_u = K_{\phi^*\omega} = \phi^* K_\omega = \phi^* K_u$ for any $\omega \in \Omega_u$ and $\phi \in \mathrm{Diff}_u(M)$. □

Let $Aut(H^2(M;\mathbb{R}))$ be the group of automorphisms of $H^2(M;\mathbb{R})$ that preserve the intersection form. For any pair of classes (a,b) in $H^2(M;\mathbb{R})$, let $D_{(a,b)}$ be the subgroup of $Aut(H^2(M;\mathbb{R}))$ that preserves the classes a and b.

We have the inclusions $\mathrm{Diff}_0(M) \subset \mathrm{Diff}_h(M) \subset \mathrm{Diff}_u(M)$ as subgroups of $\mathrm{Diff}(M)$, where $\mathrm{Diff}_h(M)$ is the subgroup of homological trivial diffeomorphisms. Note also that $\mathrm{Diff}_0(M)$ is a normal subgroup of $\mathrm{Diff}_h(M)$ since it's the identity component of a Lie group, and $\mathrm{Diff}_h(M)$ is a normal subgroup of $\mathrm{Diff}_u(M)$ since it's the kernel of the homological action of $\mathrm{Diff}_u(M)$. Let $Symp_h(M,\omega)$ be the subgroup of homologically trivial symplectomorphisms. Similarly, we have the inclusions $Symp_0(M,\omega) \subset Symp_h(M,\omega)$ as a normal subgroup and $Symp_h(M,\omega) \subset Symp(M,\omega)$ as a normal subgroup. Clearly, for any $\omega \in \Omega_u$, $\mathrm{Diff}_u(M) \supset Symp(M,\omega)$ and hence the homological action of $\mathrm{Diff}_u(M)$, which admits the injection $\mathrm{Diff}_u(M)/\mathrm{Diff}_h(M) \to Aut(H^2(M;\mathbb{R}))$, contains that of $Symp(M,\omega)$ which is the injection $Symp(M,\omega)/Symp_h(M,\omega) \to Aut(H^2(M;\mathbb{R}))$. We will show that the converse inclusion also holds.

Lemma 2.2. *Suppose M is a rational or ruled surface. Then, for $u \in \mathcal{C}_M$ and any $\omega \in \Omega_u$, the homological actions of $\mathit{Diff}_u(M)$ and $Symp(M,\omega)$ are both given by $D_{(K_u,u)}$. In other words,*

$$\mathit{Diff}_u(M)/\mathit{Diff}_h(M) = D_{(K_u,u)} = Symp(M,\omega)/Symp_h(M,\omega)$$

Proof. By Theorem 1.4 and Proposition 4.14 in [22], the homological action of $Symp(M,\omega)$ is the group $D_{(K_\omega,[\omega])}$, where $D_{(K_\omega,[\omega])}$ is the subgroup of $Aut(H^2(M;\mathbb{R}))$ that preserves the classes K_ω and $[\omega]$.

Since $Symp(M,\omega) \subset \mathrm{Diff}_u(M)$, the homological action of $\mathrm{Diff}_u(M)$ contains $D_{(K_\omega,[\omega])}$. On the other hand, the homological action of $\mathrm{Diff}_u(M)$ is contained in $D_{(K_\omega,[\omega])}$ since $\mathrm{Diff}_u(M)$ preserves $u = [\omega]$ by definition and preserves $K_u = K_\omega$ by Lemma 2.1. □

2.2. **Properties of $\mathcal{S}_u$.** Recall that $\mathcal{S}_u = \cup_{\omega \in \Omega_u} \mathcal{S}_\omega$. Here are a couple of general properties.

Lemma 2.3. *For each M and $\phi \in Diff^+(M)$, we have $\mathcal{S}_{\phi^* u} = \phi_* \mathcal{S}_u$.*

Suppose $b^+(M) = 1$ and $S \in \mathcal{S}_u$ for some $u \in \mathcal{C}_M$. Then $S \in \mathcal{S}_{u'}$ for any $u' \in \mathcal{C}_M$ pairing positively with S.

Proof. The first claim follows from $\mathcal{S}_{\phi^* \omega} = \phi_* \mathcal{S}_\omega$ and $\Omega_{\phi^* u} = \phi^* \Omega_u$.

The second claim is a direct consequence of Theorem 2.7 in [9]. □

And it is clear that $\mathcal{S}_u \supset \mathcal{S}_\omega$ for any M and $\omega \in \Omega_u$. We will show that the reverse inclusion also holds for a rational or ruled surface.

Lemma 2.4. *Let M be a rational or ruled surface. Then $\mathcal{S}_u = \mathcal{S}_\omega$ for any $\omega \in \Omega_u$.*

Proof. We just need to show $\mathcal{S}_u \subset \mathcal{S}_\omega$. Suppose $S \in \mathcal{S}_u$. Then there is an $\omega' \in \Omega_u$ and an ω'-symplectic sphere C with $S = [C]$. There is $\phi \in \mathrm{Diff}_u(M)$ so that $\phi^* \omega = \omega'$ by the transitive action of $\mathrm{Diff}_u(M)$ on Ω_u. Then the image $\phi(C)$ is an ω-symplectic sphere. By Lemma 2.2, there is a $\psi \in Symp(M, \omega)$ with the inverse homological action of ϕ. Clearly, $\psi(\phi(C))$ is an $\omega-$symplectic sphere in the class $[C]$. □

Proposition 2.5. *Let M be a rational or ruled surface.*

- *If $\mathcal{S}_u^{-1} = \mathcal{S}_{u'}^{-1}$, then $K_u = K_{u'}$.*
- *If, in addition, $\mathcal{S}_u^{-2} = \mathcal{S}_{u'}^{-2}$, then $Diff_u(M) = Diff_{u'}(M)$.*

Proof. The first statement follows from [21] so we recall some notions and facts there. Introduce the set

$$\mathcal{E} = \{E \in H_2(M, \mathbb{Z}) | E \cdot E = -1 \text{ and } E \text{ is represented by a smooth sphere}\},$$

the subset of K-exceptional spherical classes

$$\mathcal{E}_K = \{E \in \mathcal{E} | E \cdot K = -1\},$$

and the subset of ω-exceptional spherical classes

$$\mathcal{E}_\omega = \{E \in \mathcal{E} | \ E \text{ is represented by an } \omega\text{−symplectic sphere}\}.$$

By the adjunction formula, $\mathcal{E}_\omega \subset \mathcal{E}_{K_\omega}$. Here are two facts about $\mathcal{E}_K$: by Lemma 3.5 in [21], we have

$$\mathcal{E}_\omega = \mathcal{E}_{K_\omega}, \tag{3}$$

and by Theorem 4 in [21], we have

$$\mathcal{C}_{M,K} = \{e \in \mathcal{P}_M | e \cdot E > 0 \text{ for all } E \in \mathcal{E}_K\}. \tag{4}$$

Notice that $\mathcal{E}_\omega = \mathcal{S}_\omega^{-1}$ by definition. So we have $\mathcal{E}_{K_u} = S_u^{-1}$ by Lemma 2.4 and (3). It follows from (4) that $\mathcal{C}_{M,K_u}$ is determined by S_u^{-1}. Namely, if $\mathcal{S}_u^{-1} = \mathcal{S}_{u'}^{-1}$ then $\mathcal{C}_{M,K_u} = \mathcal{C}_{M,K_{u'}}$. Therefore $K_u = K_{u'}$ by Proposition 4.1 in [21].

Now we prove the second statement. By Lemma 2.2, we have

$$\mathrm{Diff}_u(M)/\mathrm{Diff}_h(M) = D_{(K_u,u)} \quad \text{and} \quad \mathrm{Diff}_{u'}(M)/\mathrm{Diff}_h(M) = D_{(K_{u'},u')}.$$

So we just need to identify $D_{(K_u,u)}$ and $D_{(K_{u'},u')}$. For this purpose, we next recall some notions and facts in [22]. Introduce the set

$$\mathcal{L} = \{L \in H_2(M; \mathbb{Z}) | L \cdot L = -2 \text{ and } L \text{ is represented by a smooth sphere}\},$$

the subset of $K-$null spherical classes

$$\mathcal{L}_K = \{L \in \mathcal{L} | L \cdot K = 0\},$$

and the subset of $(K, \alpha)-$null spherical classes

$$\mathcal{L}_{K,\alpha} = \{L \in \mathcal{L}_K | \alpha \cdot L = 0\}$$

for a class $\alpha \in \mathcal{C}_{M,K}$. By Theorem 4.14 of [22], $D_{(K,\alpha)}$ is generated by the reflections along elements in $\mathcal{L}_{K,\alpha}$. So it suffices to show that $\mathcal{L}_{K_u,u} = \mathcal{L}_{K_{u'},u'}$.

By Proposition 5.16 in [10], for a symplectic form ω and $A \in \mathcal{L}_{K_\omega}$, $A \in \mathcal{S}_\omega^{-2}$ if and only if $[\omega]$ pairs positively with A. Since $\mathcal{S}_\omega = \mathcal{S}_{[\omega]}$ by Lemma 2.4, we have the disjoint union decompositions

$$\mathcal{L}_{K_u} = \mathcal{S}_u^{-2} \coprod (-\mathcal{S}_u^{-2}) \coprod \mathcal{L}_{K_u,u}, \qquad \mathcal{L}_{K_{u'}} = \mathcal{S}_{u'}^{-2} \coprod (-\mathcal{S}_{u'}^{-2}) \coprod \mathcal{L}_{K_{u'},u'}. \tag{5}$$

Our assumptions are $\mathcal{S}_u^{-1} = \mathcal{S}_{u'}^{-1}$ and $\mathcal{S}_u^{-2} = \mathcal{S}_{u'}^{-2}$. So $K_u = K_{u'}$ by the first statement and hence $\mathcal{L}_{K_u} = \mathcal{L}_{K'_u}$. Together with the assumption $\mathcal{S}_u^{-2} = \mathcal{S}_{u'}^{-2}$ and the decompositions (5), we have $\mathcal{L}_{K_u,u} = \mathcal{L}_{K'_u,u'}$.

The proof of Lemma 2.5 is finished. □

2.3. **Level 2 chambers and the normalized reduced symplectic cone.** The following definition is partially motivated by Proposition 2.5.

Definition 2.6. *A level 2 (stability) chamber of $\mathcal{C}_M$ is a maximal path-connected subset with the same $\mathcal{S}_u^{-2}$.*

As previously remarked, for symplectic forms in the same level 2 chamber, Theorem 1.1 says that the symplectomorphism groups have the same π_0 and π_1.

When $\chi \le 12$, these level 2 chambers are best visualized by the normalized reduced symplectic cone. So we recall the notions of reduced symplectic class and normalized reduced symplectic cone, which will be also useful for the proof of Theorem 1.3. (For more details see Section 2.2 in [16]).

For $M_k = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ with the homology/cohomology basis $\{H, E_1, E_2, \cdots, E_k\}$, a class $\nu H - \sum_{i=1}^k c_i E_i \in H^2(M;\mathbb{R})$ is called **reduced** (with respect to the basis) if

$$c_1 \ge c_2 \ge \cdots \ge c_k > 0 \quad \text{and} \quad \nu \ge c_1 + c_2 + c_3.$$

A reduced symplectic class is a reduced class in $\mathcal{C}_M$. The normalized reduced symplectic cone $P_k = P(M_k) \subset \mathcal{C}_{M_k}$ is the subspace of reduced symplectic classes with $\nu = 1$. We represent such a class by $(1|c_1, \cdots, c_k)$, or $(c_1, \cdots, c_k) \in \mathbb{R}^k$.

For $\tilde{M}_1 = S^2 \times S^2$ with the basis $\{F_1 = [S^2 \times pt], F_2 = [pt \times S^2]\}$, a class $bB + fF \in \mathcal{C}_{\tilde{M}_1}$ is called reduced if $f \ge b \ge 0$. The normalized reduced symplectic cone $P_{\tilde{1}} = P(\tilde{M}_1) \subset \mathcal{C}_{\tilde{M}_1}$ is the subspace of reduced symplectic classes with $b = 1$.

We summarize properties of the normalized reduced symplectic cone $P(M)$ in the next 2 propositions.

Proposition 2.7. *Let M be a rational surface. Then*

(i) A convex combination of (normalized) reduced classes is (normalized) reduced.

(ii) A reduced class is symplectic if and only if it has positive square.

(iii) For a reduced symplectic class u, its canonical class K_u is

$$K_0 := -3H + \sum_{i=1}^k E_i$$

if $M = M_k$, and it is $K_0 := -2F_1 - 2F_2$ if $M = \tilde{M}_1$.

(iv) Every class in $\mathcal{C}_M$ is equivalent to a unique reduced symplectic class under the action of $\mathrm{Diff}^+(\mathrm{M})$. *In other words, up to scaling, the normalized reduced symplectic cone $P(M)$ is a fundamental domain of on $\mathcal{C}_M$ under the action of* $\mathrm{Diff}^+(\mathrm{M})$.

Proof. Part (i) follows directly from the definition, part (ii) is contained in [15], part (iii) is contained in [21], and part (iv) is from [21] and [31] for rational classes, and [11] for real classes (see also the Math Review of [11]). □

Definition 2.8. *For a rational surface M and K_0 as in Proposition 2.7, let*

$$\mathcal{S}_{K_0} = \cup_{u \in P(M)} \mathcal{S}_u.$$

Proposition 2.9. *Suppose M is a rational surface with $\chi \le 12$. Then*

(i) The normalized reduced cone $P(M)$ is a convex region in $\mathbb{R}^{\chi-3}$. Moreover, $-\frac{1}{3}K_0$ is in the closure.

(ii) For $M = M_k, 3 \le k \le 8$, $P(M_k)$ *is a polyhedral cone in* $\mathbb{R}^k$ *with the top vertex at* $T_k = (\frac{1}{3}, \cdots, \frac{1}{3})$ *and the convex base in the* $c_1c_2\cdots c_{k-1}$ *(i.e.* $c_k = 0$*) hyperplane generated by the following* k *points* G_i*:*

$$G_1 = (0, \cdots, 0), G_2 = (1, 0, \cdots, 0), G_3 = (\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0),$$

$$G_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \cdots, 0), \cdots, G_k = (\frac{1}{3}, \cdots, \frac{1}{3}, 0).$$

When $k = 9$, $P(M_9)$ *has a similar description except that the top vertex* $T_k = -\frac{1}{3}K_0$ *is not in* $P(M_9)$.

(iii) When $3 \le k \le 9$, *let*

$$l_1 = H - E_1 - E_2 - E_3, \quad l_2 = E_1 - E_2, \quad \cdots \quad , \quad l_k = E_{k-1} - E_k. \tag{6}$$

Then the symplectic classes on each edge T_kG_i *are characterized by the property of pairing trivially with* l_j *for any* $j \ne i$ *and positively on* l_i*. Consequently, the reduced symplectic classes are characterized as the symplectic classes which are positive on each* E_i *and non-negative on each* l_i*.*

Proof. All the statements can essentially be found in Proposition 2.21 and Section 2.2.5 in [16].

What is not explicitly covered in [16] is when $M = M_9$. In this case, we use the same argument as in Proposition 2.21 in [16], together with the following observation: For $M = M_9$, a reduced class still has non-negative square, and a normalized reduced class has square 0 if and only if it is $T_9 = -\frac{1}{3}K_0$. Hence by parts (ii) and (ii) of Proposition 2.7, the normalized reduced symplectic cone $P(M_9)$ is convex and contains all the normalized reduced classes except $T_k = -\frac{1}{3}K_0$

First, rewrite the normalized reduced condition as

$$1 \ge c_1 + c_2 + c_3, \quad c_1 \ge c_2, \quad c_2 \ge c_3, \quad \cdots, \quad c_{k-1} \ge c_9, \quad c_9 > 0. \tag{7}$$

Let Ψ be the translation moving $T_9 = (\frac{1}{3}, \cdots, \frac{1}{3})$ to 0. Under this linear translation, $(1|c_1, \cdots, c_9)$ is moved to $x = (x_1, \cdots, x_9) = (c_1 - \frac{1}{3}, \cdots, c_9 - \frac{1}{3})$, and the normalized reduced condition (7) can be written as the 9 homogeneous conditions:

$$0 \ge x_1 + x_2 + x_3, \quad x_1 - x_2 \ge 0, \quad x_2 - x_3 \ge 0, \quad \cdots, \quad x_8 - x_9 \ge 0, \quad x_9 > -\frac{1}{3}.$$

Clearly, $\Psi(P(M_9))$ has only one vertex at the origin and its opposite face is open and at the hyperplane $x_9 = -\frac{1}{3}$. There are 9 inequalities of the form $\ge$ in (7). Setting $c_9 = 0$ and all of the 9 inequality $\ge$ to be equality except the i-th one, we obtain the 9 points G_i in the $c_1\cdots c_8$ hyperplane. The rays T_9G_i are clearly extremal rays.

Notice that T_9 pairs trivially with each l_j, and G_i pairs trivially with each l_j for each $j \ne i$. It follows that T_9G_i pairs trivially with each l_j except for $j = i$. □

Finally, we point out the promised connection between level 2 chambers and $P(M)$.

Lemma 2.10. *Suppose* M *is a rational surface. Then every level 2 chamber is equivalent to a unique level 2 chamber inside* $P(M)$*.*

If $\chi(M) \le 12$, *then the level* 2 *chambers in* $P(M)$ *are precisely the open faces of* $P(M)$ *of various dimensions. In particular, they are cut out by finitely many linear inequalities.*

Proof. The first statement follows from Proposition 2.7.

The second statement follows from part (iii) of Proposition 2.9. □

3. The level ∞ chambers and $\mathcal{S}_{K_0}^{\le -3}$

Similarly, we define the level ∞ (stability) chamber as follows:

Definition 3.1. *A level* ∞ *(stability) chamber of* $\mathcal{C}_M$ *is a maximal path-connected subset with the same* $\mathcal{S}_u$*.*

Here is a simple observation.

Lemma 3.2. *Suppose M is a rational surface. Then every level ∞ chamber is equivalent to a unique level ∞ chamber inside $P(M)$.*

If $\chi(M) \le 12$, then for any $S \in \mathcal{S}_{K_0}^{\le -3}$, the hyperplane in $H^2(M;\mathbb{R})$ which vanishes on S cuts the normalized reduced symplectic cone $P(M)$ into two non-empty pieces.

Proof. The first statement again follows from Proposition 2.7.

If $S \in \mathcal{S}_{K_0}^{\le -3}$, then $-2 = S\cdot S + S\cdot K_0$ by the adjunction formula. Hence

$$S\cdot(-K_0) = 2 + S\cdot S < 0.$$

On the other hand, by the definition of $\mathcal{S}_{K_0}$, $S \in \mathcal{S}_u$ for some $u \in P(M)$. For such a class u, we must have $u\cdot S > 0$. By Proposition 2.9, $T_k = \frac{1}{3}K_0$ is in the closure of $P(M)$. Then it is clear that the hyperplane defined by S divides $P(M)$ into two non-empty pieces. □

3.1. **Finiteness of $\mathcal{S}_u^{\le -3}$.** The following finiteness result will be useful.

Lemma 3.3. *Suppose M is a rational surface with $\chi(M) \le 12$. Then $\mathcal{S}_u^{\le -3}$ is a finite set for any $u \in \mathcal{C}_M$.*

Proof. By Proposition 2.7 and Lemma 2.3 we can assume that u is reduced (and normalized).

This is easy for $M = S^2 \times S^2$.

So we assume that $M = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}, n \le 9$. Let

$$S = aH + \sum b_i E_i \in \mathcal{S}_u^{\le -2}.$$

We have the adjunction formula

$$a^2 - 3a + 2 = \sum b_i^2 + \sum b_i. \tag{8}$$

We first observe that for a fixed a, there are only finitely choices of vectors $(b_1, ..., b_n)$ satisfying (8). This is true since the non-negative quadratic function $f(x) = x(x+1)$ on the set of integers $\mathbb{Z}$ has the property that the inverse image of any finite interval is a finite set.

We now discuss three cases: $a > 0$, $a = 0$ and $a < 0$.

- $a > 0$.

By the observation above, it suffices to bound a from above. We will achieve this by slightly modifying Zhang's argument in [30]. When $\chi(M) < 12$, it follows from Proposition 4.6 in [30] that the set $\{S \in \mathcal{S}_{K_0}^{\le -1} | a > 0\}$ is finite.

Let $c^2 = -S\cdot S$ for a real number $c \ge 2$. We then have

$$c^2 + a^2 = b_1^2 + \cdots b_n^2. \tag{9}$$

We rewrite (8) as

$$-2 + c^2 + 3a = b_1 + \cdots + b_n. \tag{10}$$

Applying Cauchy-Schwartz to (10), together with (9), we have

$$(c^2 + 3a - 2)^2 \le n(b_1^2 + \cdots + b_n^2) = n(c^2 + a^2) \le 9(c^2 + a^2).$$

This can be written as

$$6a(c^2 - 2) \le -(c^4 - 13c^2 + 4) = -(c^2 - \frac{13}{2})^2 + \frac{169}{4} - 4 \le 39. \tag{11}$$

If $c^2 \ge 3$, then we have $0 < a \le 7$ by (11).

Note that we actually proved that $\{S \in \mathcal{S}_{K_0}^{\le -3} | a > 0\}$ is a finite set when $\chi(M) \le 12$.

- $a = 0$

In this case, we have

$$\{S \in \mathcal{S}_{K_0}^{\le -2} | a = 0\} = \{E_m + \sum_{j=m+1}^{n} b_j E_j, \quad m \ge 1, \quad b_j = 0 \text{ or } -1\}. \tag{12}$$

In particular, $\{S \in \mathcal{S}_{K_0}^{\leq -2} | a = 0\}$ is a finite set since n is fixed.

This claim follows simply from the adjunction formula (8). Rewrite (8) as

$$2 = (a-1)(a-2) = (b_1+1)b_1 + \sum_{j=2}^{n}(b_j+1)b_j.$$

As already observed since b_i are integers, we have $(b_i+1)b_i \geq 0$ for any i. Moreover, $(b_i+1)b_i \leq 2$ only if $b_i = -2, -1, 0, 1$ and $(b_i+1)b_i = 2$ when $b_i = -2$ or 1. But if $b_{i_0} = -2$ for some i_0, then all other $b_i = 0$. But this is impossible since $u \cdot (-2E_{j_0}) < 0$.

Suppose $b_m \neq 0$ and $b_i = 0$ for any $i < m$. Then $b_m = -1$ or 1. If $b_m = 1$, then $b_j = 0, -1$ for $j > m$ since $(b_m+1)b_m = 2$.

We exclude the case $b_m = -1$. If $b_m = -1$, since $u \cdot S > 0$, we must have $b_{j_0} > 0$ for some ${}_0j > m$. But there can only one such j_0. However, since u is reduced, $u(E_m) \geq u(E_{j_0})$ and hence $u \cdot E \leq 0$. This is impossible.

- $a < 0$

In this case, it follows from Lemma 3.2 in [8] that

$$\{S \in \mathcal{S}_{K_0}^{\leq -2} | a < 0\} = \{-pH + (p+1)E_1 + \sum_{j=2}^{n} b_j E_j, \quad p < 0, \quad b_j = 0 \text{ or } -1\}. \tag{13}$$

Write

$$u = H - c_1 E_1 - \sum_{i=2}^{n} c_i E_i$$

with $0 < c_i < c_1$. Then for $p > 0, b_j = 0$ or -1 for $j \geq 2$ and

$$S = -pH + (p+1)E_1 + \sum_{j=2}^{n} b_j E_j \in \mathcal{S}_u^{\leq -2} \subset \mathcal{S}_{K_0}^{\leq -2},$$

since $u \cdot E_j > 0$, we have

$$\begin{aligned} u \cdot S &= u \cdot (-pH + (p+1)E_1) + u \cdot (\sum_{j=2}^{n} b_j E_j) \\ &\leq u \cdot (-pH + (p+1)E_1) \\ &= -p + (p+1)c_1 \\ &\leq 0 \end{aligned}$$

if p is sufficiently large. It follows that $\{S \in \mathcal{S}_u^{\leq -2} | a < 0\}$ is a finite set. $\square$

We call a point in $\mathcal{C}_M$ a **rational point** if every coordinate of the point is rational, otherwise we call it an irrational point.

Corollary 3.4. *For a rational surface with $\chi \leq 12$, each level ∞ chamber is a convex region defined by a finite set of linear inequalities.*

Except for the monotone chamber which consists of the single point $-K_0$, each level ∞ chamber is of positive dimension.

In particular, there are infinitely many (indeed dense) rational points and irrational points.

Proof. The first statement directly follows from Lemma 2.10 and Lemma 3.3, which says respectively that each level 2 chamber is a convex region cut by finitely many linear inequalities and that there are only finitely many elements in $\mathcal{S}_u^{\leq -3}$.

The second statement is a corollary of the first one, since any level ∞ chamber is obtained by cutting the interior or a facet of the reduced cone by finitely many hyperplanes. $\square$

Remark 3.5. *We mention a couple of facts that are not needed for the proof of Theorem 1.3. The set $\mathcal{S}_u^{-2}$ is finite if $\chi(M) \leq 11$. This is derived when $a = 0$ and $a < 0$ in the proof of Lemma 3.3. When $a > 0$, it is directly argued as follows. If $c^2 = 2$ and $M = M_n, n \leq 8$, then by adding $b_i = 0$ for $n+1 \leq i \leq 9$, we have*

$$3a = \sum_{i=1}^{9} b_i, \quad a^2 - \sum_{i=1}^{9} b_i^2 = -2.$$

Note that $b_9 = 0$ and $(a - 3b_i)^2 \leq 18$ for each i, so we have $a \leq 4$.

However, the set $\mathcal{S}_{K_0}^{-2}$ is actually infinite when $\chi(M) = 12$.

3.2. **The level ∞ chambers for small χ.** We explicitly draw the reduced cone and the level ∞ stable chambers when the Euler number is small.

3.2.1. M_3. In this case, the homotopy groups of $Symp(M, \omega)$ are calculated in [5] Proposition 3.3. The level ∞ chambers can be read off and agree with Theorem 1.1 in this case.

Here the polytope OAM_2M_3 is the reduced symplectic cone of $\mathbb{C}P^2 \# 3\overline{\mathbb{C}P^2}$. The hyperplanes in the interior of the cone are the walls defined by the curves in classes $S^{\leq -2}$.

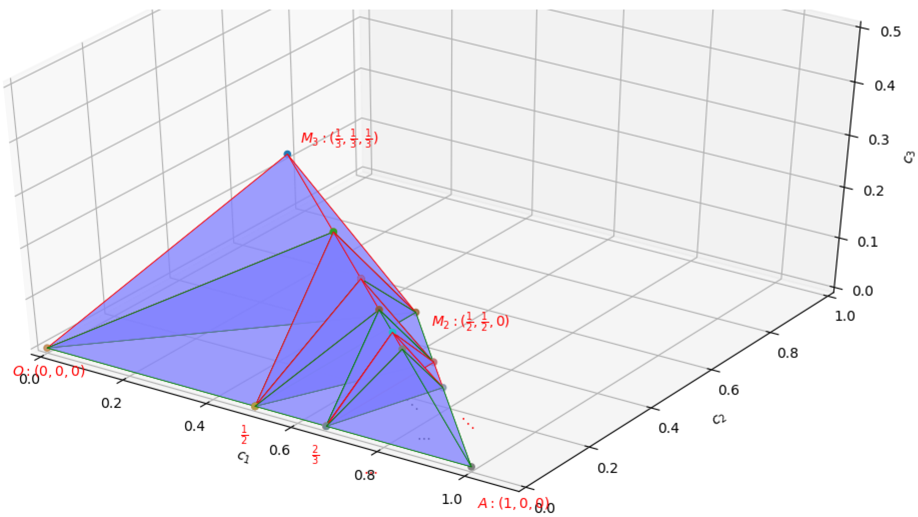

FIGURE 1. stable chambers of 3-point blowup

Here we list the classes of the interior walls: the first wall from the left is defined by the curve $E_1 - E_2 - E_3$; then as one moving right toward point A, for each integer $k > 1$ there's a pattern of 4 walls

$$kE_1 - (k-1)H,\ kE_1 - (k-1)H - E_3,\ kE_1 - (k-1)H - E_2,\ kE_1 - (k-1)H - E_2 - E_3.$$

They never intersect with each other except on the edges. Those walls also belong to the family of curves in the proof of Lemma 3.3.

3.2.2. M_2. In this case, the level ∞ chambers can be read off from Theorems 1.1 and 1.6 in [14], where the homotopy groups are calculated.

The c_1c_2 plane slice of the cone in Figure 1 is the cone of the 2-point blow up. We illustrate it in Figure 2.

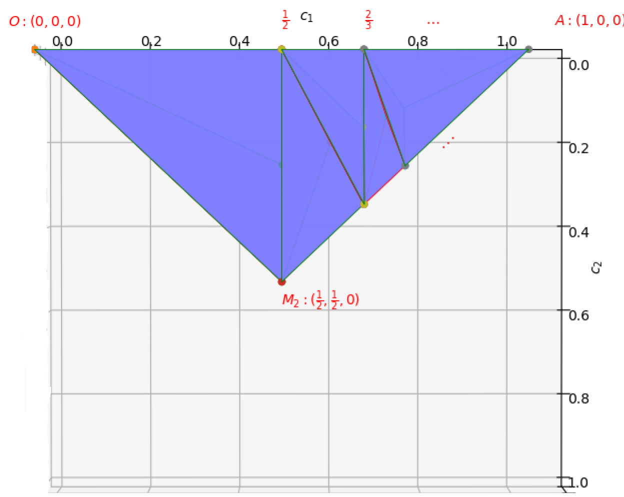

FIGURE 2. stable chambers of 2-point blowup

Note that the above walls (the lines in the interior of the triangle) come in pairs

$$2E_1 - H, 2E_1 - H - E_2, \quad 3E_1 - 2H, 3E_1 - 2H - E_2, \quad \cdots \quad , kE_1 - (k-1)H, kE_1 - (k-1)H - E_2, \quad \cdots.$$

They all belong to the family of curves in the proof of Lemma 3.3. They divide the reduced cone into level ∞ chambers.

4. The decomposition of $\mathcal{A}_u$, the Almost Kähler cone and J–inflation

We now turn to almost complex structures.

4.1. **The homotopy fibration** $Symp(M,\omega) \to \mathbf{Diff}_u(M) \to \mathcal{A}_u$**.** For $u \in \mathcal{C}_M$, recall we introduced $\mathcal{A}_u$, the space of almost complex structures that are compatible with some $\omega \in \Omega_u$.

Both Ω_u and $\mathcal{A}_u$ are infinite dimensional Fréchet manifolds. The following observation is proved by the argument in [24] for the pair of the space Ω_ω of symplectic forms that are isotopic to a symplectic form ω and the space of almost complex structures tamed by some form in Ω_ω.

Lemma 4.1. *Ω_u is canonically homotopy equivalent to $\mathcal{A}_u$. In particular, there is a canonical bijection between the sets of path connected components of Ω_u and $\mathcal{A}_u$. Moreover, $Diff_u(M)/Diff_0(M)$ acts transitively on the sets of path connected components of Ω_u and $\mathcal{A}_u$.*

Proof. Consider the space P_u of pairs

$$P_u = \{(\omega, J) \in \Omega_u \times \mathcal{A}_u | \omega \text{ is compatible with } J\},$$

Since the projection $\alpha_u : P_u \to \mathcal{A}_u$ is a fibration with the fiber at J being the convex set of J-compatible symplectic forms, the projection is a homotopy equivalence. The projection $\beta_u : P_u \to \Omega_u$ is also a homotopy equivalence since it is a fibration with the fiber at ω being the contractible set of ω-compatible almost complex structures. Let $\gamma_u : \Omega_u \to P_u$ be a homotopy inverse of β_u. Then $\alpha_u \circ \gamma_u$ is the desired canonical homotopy equivalence between Ω_u and $\mathcal{A}_u$.

Of course, $\alpha_u \circ \gamma_u$ also induces a canonical bijection between the sets of connected components of Ω_u and $\mathcal{A}_u$. Since $\mathrm{Diff}_u(M)$ acts transitively on Ω_u, there is a faithful transitive action of $\mathrm{Diff}_u(M)/\mathrm{Diff}_0(M)$ on the the sets of path connected components of Ω_u and $\mathcal{A}_u$. $\square$

Via the homotopy equivalence and the fibration (2), we arrive at the following homotopy fibration:

$$Symp(M,\omega) \to \mathrm{Diff}_u(M) \to \mathcal{A}_u. \tag{14}$$

To apply this homotopy fibration to $\pi_i(Symp(M,\omega))$, we next introduce a decomposition of $\mathcal{A}_u$ via $\mathcal{S}_u$.

4.2. **A disjoint union decomposition of $\mathcal{A}_u$ via $\mathcal{S}_u$.** When (X,ω) is a symplectic 4-manifold, we introduced in [16] a decomposition of $\mathcal{J}_\omega$ via embedded ω-symplectic spheres of self-intersection at most -2. We introduce the corresponding decomposition for $\mathcal{A}_u$.

For each $A \in \mathcal{S}_u^{<0}$ we associate the non-negative even integer

$$cod_A = 2(-A\cdot A - 1).$$

Definition 4.2. *Let u be a symplectic class. Given a finite subset $\mathcal{C} \subset \mathcal{S}_u^{<0}$,*

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0 \ \textit{if}\ i \neq j\},$$

define the codimension of the set $\mathcal{C}$ as $cod(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} cod_{A_i}$.

We call such a set $\mathcal{C}$ an **admissible subset of $S_u^{<0}$**.

Notice that $cod(\mathcal{C})$ is a non-negative integer and $cod(\mathcal{C}) \leq cod(\mathcal{C}')$ if $\mathcal{C} \subset \mathcal{C}'$.

Definition 4.3. *Given $\mathcal{C}$ as above, we define* **prime subsets**

$$\mathcal{A}_{u,\mathcal{C}} = \{J \in \mathcal{A}_u | A \in \mathcal{S}_u \ \textit{has an embedded } J\textit{-hol representative } \textbf{if and only if } A \in \mathcal{C}\}.$$

And we define $cod(\mathcal{A}_{u,\mathcal{C}}) = cod(\mathcal{C})$.

Clearly, we have the disjoint union decomposition: $\mathcal{A}_u = \amalg_{\mathcal{C}} \mathcal{A}_{u,\mathcal{C}}$.

We introduce the following filtration $\{\mathcal{A}_u^{2n}\}$ of $\mathcal{A}_u$, with

$$\mathcal{A}_u^{2n} = \amalg_{\mathcal{C}, cod(\mathcal{C})<2n} \mathcal{A}_{u,\mathcal{C}}. \tag{15}$$

Let $\mathcal{X}_{u,2n}$ denote the complement of $\mathcal{A}_u^{2n}$. Clearly, it can also be written as a **disjoint** union

$$\mathcal{X}_{u,2n} = \amalg_{\mathcal{C}, cod(\mathcal{C})\geq 2n} \mathcal{A}_{u,\mathcal{C}}. \tag{16}$$

It is actually useful to express $\mathcal{X}_{u,2n}$ as the following (not necessarily disjoint) union.

$$\mathcal{X}_{u,2n} = \cup_{\mathcal{C}, cod(\mathcal{C})\geq 2n} U_{u,\mathcal{C}}, \tag{17}$$

where

$$U_{u,\mathcal{C}} = \{J \in \mathcal{A}_u | A \in \mathcal{S}_u \text{ has an embedded } J\text{-hol representative } \textbf{if } A \in \mathcal{C}\}.$$

Clearly, $U_{u,\mathcal{C}} \supset \mathcal{A}_{u,\mathcal{C}}$ and $U_{u,\mathcal{C}} = \amalg_{\mathcal{C}', \mathcal{C}' \supset \mathcal{C}} \mathcal{A}_{u,\mathcal{C}'}$

We have the following analogue of [5] Proposition B.1,

Proposition 4.4. *Each subset $U_{u,\mathcal{C}}$ is a co-oriented Fréchet submanifold of $\mathcal{A}_u$ of (real) codimension $cod(\mathcal{C})$. It follows that $\mathcal{X}_{u,2n}$ is the union of submanifolds with codimension at least $2n$.*

The only difference is that we consider the space $\mathcal{A}_u$ instead of $\mathcal{J}_\omega$ here. And the proof is the same, considering the projection from the universal moduli of curves in classes $\mathcal{C}$ onto $\mathcal{A}_u$ and compute the index of the linearized operator. (See also [1] appendix and [24] Lemma 2.6). The index computation goes the same as Proposition B.1 of [5].

Remark 4.5. *The analogue of Proposition 2.14 in [16] is also valid for $\mathcal{A}_{u,\mathcal{C}}$: when $\chi \leq 12$, $\mathcal{A}_{u,\mathcal{C}}$ are submanifolds of $\mathcal{A}_u$ with codimension $cod(\mathcal{C})$. The proof is also similar.*

4.3. **The almost Kähler cone.** Recall the two notions of J-symplectic cones, the J-tame cone and the J-compatible cone:

$$\begin{array}{rcl} \mathcal{K}_J^t & = & \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ tames } J\}, \\ \mathcal{K}_J^c & = & \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ is compatible with } J\}. \end{array} \tag{18}$$

$\mathcal{K}_J^c$ is also called the almost Kähler cone. Both $\mathcal{K}_J^c$ and $\mathcal{K}_J^t$ are convex cohomology cones contained in the positive cone $\mathcal{P}_M = \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\}$.

Clearly, $\mathcal{K}_J^c \subset \mathcal{K}_J^t$. And for an almost Kähler J on a 4-manifold with $b^+ = 1$, they are equal.

Theorem 4.6 (Theorem 1.3 in [23])**.** *Let (M, J) be an almost complex 4-manifold. If $b^+(M) = 1$ and $K_J^c \neq \emptyset$, then $\mathcal{K}_J^c = \mathcal{K}_J^t$.*

Let $C_J(M)$ be the curve cone of a compatible almost complex manifold (M, J):

$$C_J(M) = \{\sum a_i [C_i] | a_i > 0,\ C_i \text{ is a irreducible J-holomorphic subvariety }\}.$$

Let $C_J^{\vee,>0}(M)$ be the positive dual of $C_J(M)$ under the homology-cohomology pairing, and set

$$\mathcal{P}_J = C_J^{\vee,>0}(M) \cap \mathcal{P}.$$

Clearly, $\mathcal{K}_J^t \subset C_J^{\vee,>0}(M)$ since the integral of a J-tamed symplectic form over a J-holomorphic subvariety is positive. Motivated by the famous Nakai-Moishezon-Kleiman criterion in algebraic geometry which characterizes the ample cone in terms of the (closure of) curve cone for a projective J, and the recent Kähler version[1] of the Nakai-Moishezon criterion (in dimension 4), which characterizes the Kähler cone in terms of the curve cone for a Kähler J, one asks whether there is a tamed/almost Kähler version of the Nakai-Moishezon criterion.

Here is the almost Kähler Nakai-Moishezon theorem for rational surfaces with $\chi \leq 12$:

Theorem 4.7 (Theorem 1.6 in [30])**.** *Suppose M is a rational surface with $\chi \leq 12$. For an almost Kähler J on M, the dual cone of the curve cone is the almost Kähler cone, i.e. $C_J^{\vee,>0}(M) = K_J^c(M)$.*

4.4. **A remark on applying J-inflations to 4-manifolds with $b^+ = 1$.** An important ingredient for Theorem 4.7 is the tamed J-inflation by McDuff [24] and Buse [6]. Note that the proofs make the unwarranted assumption that for every ω-tame J and every J-holomorphic curve Z one can find a family of normal planes that is both J invariant and ω-orthogonal to TZ. This is true only if ω is compatible with J at every point of Z. We state here a weaker version of Lemma 3.1 in [24] and Theorem 1.1 in [6]:

Given a compatible pair (J, ω), one can inflate along a J-holomorphic curve Z, so that there exist a symplectic form ω' taming J such that $[\omega'] = [\omega] + tPD(Z), t \in [0, \lambda)$ where $\lambda = \infty$ if $Z \cdot Z \geq 0$ and $\lambda = \frac{\omega(Z)}{(-Z \cdot Z)}$ if $Z \cdot Z < 0$.

For M with $b^+ = 1$ and a compatible pair (J, ω) on M, $\mathcal{K}_J^c = \mathcal{K}_J^t$ by Theorem 4.6. So if we start with this almost Kähler J and perform the above inflation to obtain a J-tame ω' in the cohomology class $[\omega] + tPD(Z)$, then we have a $J-$compatible ω'' cohomologous to ω' by the equality between the $J-$tame cone and the $J-$compatible cone. We call this process the $b^+ = 1$ $J-$**compatible inflation**.

This $b^+ = 1$ $J-$compatible inflation is sufficient for the proof of Theorem 4.7. It is also sufficient for all the known stability results of $Symp(M, w)$ when $b^+(M) = 1$ (see Section 5.2).

Remark 4.8. *Recently, Chakravarthy and Pinsonnault were able to restore the tamed version when $Z \cdot Z \leq 0$ ([7]).*

5. Stability of $Symp(X, \omega)$

We prove Theorem 1.3 in this section. We also outline a related approach via the homotopy fibration (21). Finally, we connect the topology of $Symp(M, \omega)$ with the topology of the space of symplectically embedded balls.

[1] Established by Buchdahl and Lamari in dimension 4, and by Demailly-Paun in arbitrary dimension.

5.1. **Proof of Theorem 1.3.**

Lemma 5.1. *Let M be a rational surface with $\chi \leq 12$. If $\mathcal{S}_u \subset \mathcal{S}_{u'}$, then $\mathcal{A}_u \subset \mathcal{A}_{u'}$.*

Proof. Suppose $J \in \mathcal{A}_u$, ie. J is compatible with some $\omega \in \Omega_u$. Let S_J be the set of classes of embedded J-holomorphic rational curves. Then clearly $S_J \subset \mathcal{S}_\omega$ for any ω taming J.

Since w' is positive on $\mathcal{S}_{u'} = \mathcal{S}_{\omega'} \supset \mathcal{S}_u$, we have u' is positive on S_J. Then by Theorem 4.7 we conclude that u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u'}$. We have shown $\mathcal{A}_u \subset \mathcal{A}_{u'}$ □

The following lemma is the case when we have inclusion from two directions.

Lemma 5.2. *If $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}, n \geq 2$, then $\mathcal{A}_u^{2n} = \mathcal{A}_{u'}^{2n}$.*

Proof. We first observe that if $\mathcal{C} \subset S_u^{<0}$ with $cod(\mathcal{C}) < 2n$, then $\mathcal{C} \subset \mathcal{S}_u^{\geq -n}$.

Since $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$, the decompositions of $\mathcal{A}_u^{2n}$ and $\mathcal{A}_{u'}^{2n}$ in (15) are indexed by the same set of admissible subsets of $S_u^{\geq -n}$, all with codimension less than $2n$. We will show that, for each such $\mathcal{C}$, we have $\mathcal{A}_{u,\mathcal{C}} = \mathcal{A}_{u',\mathcal{C}}$.

We take any such admissible subset $\mathcal{C}$. If $J \in \mathcal{A}_{u,\mathcal{C}}$, then J is compatible with some $w \in \Omega_u$ and the **only** J-holomorphic curves are in the classes of $\mathcal{C}$. Since u' is positive on all of the classes in $\mathcal{C}$ and hence by Theorem 4.7, u' is in the almost Kähler cone of J, ie. J is compatible with some symplectic form ω' with class u'. In other words, $J \in \mathcal{A}_{u',\mathcal{C}}$. This means that $\mathcal{A}_{u,\mathcal{C}} \subset \mathcal{A}_{u',\mathcal{C}}$. The same strategy applies to prove the converse. □

To be more explicit, let us look at the case $n = 3$. Suppose $J \in \mathcal{A}_{u,A}$ where $A \in \mathcal{S}_u^{-3}$. Then J is compatible with some $w \in \Omega_u$ and the only J-holomorphic curves with self-intersection at most -3 are in the class A. Since $A \in \mathcal{S}_u^{-3} = \mathcal{S}_{u'}^{-3}$, u' is positive on A, and hence by Theorem 4.7, u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u',A}$. Converse inclusion is proved in the same way.

Suppose $J \in \mathcal{A}_{u,\{A_1,A_2\}}$ where $A_i \in \mathcal{S}_u^{-2}$. Then J is compatible with some $w \in \Omega_u$ and the **only** J-holomorphic curves with self-intersection at most -2 are in the class A_i, $i = 1, 2$. Since $A_i \in \mathcal{S}_u^{-2} = \mathcal{S}_{u'}^{-2}$, u' is positive on A_i (as well as on $\mathcal{S}_u^{-1}$) and hence by Theorem 4.7, u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u',\{A_1,A_2\}}$. The converse inclusion is proved in the same way. The case that the admissible set $\mathcal{C} = \{A\}$ where A is of square -2 also follows from the same argument.

The following is essentially due to McDuff's beautiful observation in [24] and Proposition 2.5.

Lemma 5.3. *Suppose $\mathcal{A}_u \subset \mathcal{A}_{u'}$ and $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}, n \geq 2$. Then the groups π_i of $Symp(M, u)$ and $Symp(M, u')$ are the same for $i \leq 2n - 3$.*

Proof. The inclusion map $\mathcal{A}_u \subset \mathcal{A}_{u'}$ induces a continuous map $Symp(M,\omega) \to Symp(M,\omega')$, well defined up to homotopy and connecting the sequences (14) for w and w' into the following homotopy commutative diagram:

$$\begin{array}{ccccc} Symp(M,\omega) & \longrightarrow & \mathrm{Diff}_u(M) & \longrightarrow & \mathcal{A}_u \\ \downarrow & & = \downarrow & & \downarrow \\ Symp(M,\omega') & \longrightarrow & \mathrm{Diff}_{u'}(M) & \longrightarrow & \mathcal{A}_{u'}. \end{array} \tag{19}$$

Firstly, by Proposition 2.5, $\mathrm{Diff}_u(M) = \mathrm{Diff}_{u'}(M)$ and the actions on Ω_u and $\Omega_{u'}$ are the same. Hence the above diagram of homotopy fibrations commutes.

We claim that the complement of $\mathcal{A}_u \subset \mathcal{A}_{u'}$ is a union of submanifolds of codimension higher than $2n$. Since by Lemma 5.2 they have the same prime sets up to codimension $2n - 2$, i.e $\mathcal{A}_u^{2n} = \mathcal{A}_{u'}^{2n}$, and by Lemma 4.4, $\mathcal{X}_{u,2n}$, $\mathcal{X}_{u',2n}$ each is a union of submanifolds with codimension greater than $2n$. Then by standard transversality argument, the inclusion induce an isomorphism $\pi_i(\mathcal{A}_u) \to \pi_i(\mathcal{A}_{u'})$ for $i \leq 2n-2$. Therefore, from the homotopy commuting diagram and the associated diagram of long exact sequences of homotopy groups, the induced homomorphisms $\pi_i(Symp(M, u)) \to \pi_i(Symp(M, u'))$ are isomorphisms for $i \leq 2n - 3$.

□

We now have established a weaker version of Theorem 1.3.

Proposition 5.4. *Let M be a rational surface, and u, u' be two symplectic classes. If $\mathcal{S}_u \subset \mathcal{S}_{u'}$ and $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$, for some $n \geq 2$, then the groups π_i of $Symp(M,u)$ and $Symp(M,u')$ are the same for $i \leq 2n-3$.*

Consequently, if $\mathcal{S}_u = \mathcal{S}_{u'}$, then π_i of $Symp(M,u)$ and $Symp(M,u')$ are the same for all i.

Proof. It follows from the three lemmas above, Lemma 5.3, Lemma 5.1, Lemma 5.2. □

To prove Theorem 1.3, it still remains to remove the condition $\mathcal{S}_u \subset \mathcal{S}_{u'}$. We will actually reduce the general case to the case $\mathcal{S}_u \subset \mathcal{S}_{u'}$.

Proof of Theorem 1.3. We first prove that in the level ∞ chamber, any two symplectic forms ω and ω' has $\pi_i(Symp(M,\omega)) = \pi_i(Symp(M,\omega'))$ for all $i \geq 0$. This directly follows from Proposition 5.4, since in the level ∞ chamber, ω and ω' has $\mathcal{S}_{[\omega]} = \mathcal{S}_{\omega'}$ and hence the assumptions are satisfied.

Then we deal with the first statement of Theorem 1.3. We can assume both u and u' are reduced and will apply Lemmas 3.2 and 3.3. We will use the following (standard) notation of symmetric difference of two sets A, B:

$$A \vartriangle B = (A \setminus B) \cup (B \setminus A).$$

Consider $\mathcal{S}_u \vartriangle \mathcal{S}_{u'}$. Since $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$, we have

$$\mathcal{S}_u \vartriangle \mathcal{S}_{u'} \subset \mathcal{S}_u^{\leq -(n+1)} \cup \mathcal{S}_{u'}^{\leq -(n+1)} \subset \mathcal{S}_{K_0}^{\leq -(n+1)}. \tag{20}$$

It follows from Lemma 3.3 that $\mathcal{S}_u \vartriangle \mathcal{S}_{u'}$ is a finite set $\{S_1, \cdots, S_k\}$.

By Lemma 3.2, the reduced symplectic cone is convex so it contains the line segment $\overline{uu'}$ connecting u and u'.

We call the line segment $\overline{uu'}$ is in the generic position if all the transverse intersection points with walls 1) are pairwise distinct, 2) only appear in the interior of $\overline{uu'}$.

Now let us assume that neither u nor u' is the monotone class. Then by Lemma 3.4, their level ∞ chamber is of positive dimension and hence has a non-empty interior.

If the line segment $\overline{uu'}$ is not in the generic position, then we can do a small perturbation of u and u' in the interior of their level ∞ chamber respectively, making $\overline{uu'}$ in generic position. Then reason is that by Lemma 3.3, for any level ∞ chamber other than the monotone point, there are both rational points and irrational points. Also, the intersection points of walls are all rational points, since they are solutions to a linear equation system with integer coefficients. And hence we can perturb u' to a rational point and u to an irrational point within their own level ∞ chamber, then $\overline{uu'}$ must be in generic position. (If $\overline{uu'}$ completely lies in a hyperplane or an intersection of hyperplanes, there exist such generic perturbation within such hyperplane or intersection.)

Now we have a line segment $\overline{uu'}$ in the generic position, and we can give our main argument: First of all, $\overline{uu'}$ intersects each of the hyperplane defined by an element in $\mathcal{S}_u \vartriangle \mathcal{S}_{u'}$ exactly at one point in the interval (u, u'), since $\overline{uu'}$ is in generic position. (Throughout this proof $\overline{uu'}$ means the open interval (u, u') unless otherwise specified). This follows from the definition of $\mathcal{S}_u \vartriangle \mathcal{S}_{u'}$ and Lemma 2.3.

Secondly and very importantly, the interior of $\overline{uu'}$ either lies in or does not intersect any hyperplane defined by any $S \in \mathcal{S}_{K_0}^{\leq -2} \setminus \mathcal{S}_u \vartriangle \mathcal{S}_{u'}$. This is because, by definition, we have the disjoint union

$$\mathcal{S}_{K_0} = (\mathcal{S}_u \vartriangle \mathcal{S}_{u'}) \coprod (\mathcal{S}_u \cap \mathcal{S}_{u'}) \coprod (\mathcal{S}_u \cup \mathcal{S}_{u'})^c,$$

where $(\mathcal{S}_u \cup \mathcal{S}_{u'})^c := \mathcal{S}_{K_0} \setminus (\mathcal{S}_u \cup \mathcal{S}_{u'})$.

For any class $x \in \mathcal{S}_{K_0}$, either $x \in (\mathcal{S}_u \vartriangle \mathcal{S}_{u'})$ or $x \in (\mathcal{S}_u \cap \mathcal{S}_{u'}) \cup (\mathcal{S}_u \cup \mathcal{S}_{u'})^c$. If $x \in (\mathcal{S}_u \cap \mathcal{S}_{u'})$, this means u, u' both pair x positive; and if $x \in (\mathcal{S}_u \cup \mathcal{S}_{u'})^c$ this means u, u' both pair x non-positive. By convexity, in the former case $\overline{uu'}$ lies in the hyperplane, and in the latter case, the interior of $\overline{uu'}$ does not intersect the hyperplane. If there is an intersection point in the interior of $\overline{uu'}$, then this means one of u, u' pairs with S positively and the other one doesn't. By Lemma 2.3, $S \in \mathcal{S}_u \vartriangle \mathcal{S}_{u'}$.

Then we call the intersection points p_i for each class $S_i \in \mathcal{S}_u \triangle \mathcal{S}_{u'}, i = 1, \cdots, k$. The line segment $\overline{uu'}$ is in generic position and hence those p_i's are distinct. Then there are $k+1$ intervals on $\overline{uu'}$. Pick $u_0 = u$ from the first interval, $u_k = u'$ from the last interval, and one point $u_i, 1 \le i < k$, from the interior of each of the remaining $k-1$ intervals. Then for each $0 \le i < k$, we consider the pairs u_i and u_{i+1}.

For this pair, the key observation is that $\mathcal{S}_{u_i}$ and $\mathcal{S}_{u_{i+1}}$ differ by only the class S_i. Consequently, $\mathcal{S}^{\ge -n}_{u_i} = \mathcal{S}^{\ge -n}_{u_{i+1}}$ since $S_i \in \mathcal{S}_u \triangle \mathcal{S}_{u'} \subset \mathcal{S}^{\le -(n+1)}_{K_0}$ by (20). Moreover, we either have $\mathcal{S}_{u_i} \subset \mathcal{S}_{u_{i+1}}$ or $\mathcal{S}_{u_{i+1}} \subset \mathcal{S}_{u_i}$. Without loss of generality, we assume that $\mathcal{S}_{u_i} \subset \mathcal{S}_{u_{i+1}}$. Then we have the inclusion $\mathcal{A}_{u_i} \subset \mathcal{A}_{u_{i+1}}$ by Lemma 5.1. Since $\mathcal{S}^{\ge -n}_{u_i} = \mathcal{S}^{\ge -n}_{u_{i+1}}$ as just observed, it follows that $\mathcal{A}^{2n}_{u_i} = \mathcal{A}^{2n}_{u_{i+1}}$ by Lemma 5.2. Now all the assumptions for Lemma 5.4 are satisfied and we can now conclude that $\pi_j(Symp(M, \omega_i)) = \pi_j(Symp(M, \omega_{i+1}))$ for $1 \le j \le 2n-3$, where $[\omega_i] = u_i$ and $[\omega_{i+1}] = u_{i+1}$. Then doing induction we have established the first statement of Theorem 1.3 for ω and ω'.

Then the second statement of Theorem 1.3 is a direct corollary of the first one.

□

Remark 5.5. *We speculate that Theorem 1.3 holds for arbitrary rational surfaces. However, there are difficulties to generalize the arguments to rational surfaces with $\chi \ge 13$. We mention a couple here.*
1) The almost Kähler Nakai-Moishezon criterion has not been established for these rational surfaces.
2) Lemma 3.3 is not valid for these rational surfaces and so the "walls" are somewhere dense in the symplectic cone.

5.2. **More remarks.** We first discuss a related approach via Kronheimer's homotopy fibration.

Let ω be a symplectic form on M and Ω_ω the space of symplectic forms that are isotopic to ω. If $\omega \in \Omega_u$, then Ω_ω is the path connected component of Ω_u containing ω. Let $G_\omega = Symp(M, \omega) \cap \mathrm{Diff}_0(M)$. Moser's lemma grants that $\mathrm{Diff}_0(M)$ acts transitively on Ω_ω, hence we have the following fibration in [12]:

$$G_\omega \to \mathrm{Diff}_0(M) \to \Omega_\omega. \tag{21}$$

Note that fibration (21) agrees with fibration (14) if $\Omega_\omega = \Omega_u$, i.e. Ω_u is path connected, or equivalently, Diff_u is path connected. This is possible for a generic ω. But this is unknown for any ω, even on $\mathbb{C}P^2$. Note that it is conjectured that $\Omega_{[\omega_{FS}]}$ is contractible in [26] section 14.1.

Let $\mathcal{A}_{\Omega_\omega}$ be the space of almost complex structures that are compatible with some element in Ω_ω and $P_{\Omega_\omega} = \{(\alpha, J) \in \Omega_\omega \times \mathcal{A}_{\Omega_\omega} | \alpha \text{ is compatible with } J\}$. By the argument in Lemma 4.1, $\mathcal{A}_{\Omega_\omega}$ and P_{Ω_ω} are path connected. Moreover, $\mathcal{A}_{\Omega_\omega}$ is a path connected component of $\mathcal{A}_u$ and is canonically homotopy equivalent to Ω_ω. In fact, Ω_ω and $\mathcal{A}_{\Omega_\omega}$ correspond to each other under the canonical bijection between the sets of path connected components of Ω_u and $\mathcal{A}_u$ in Lemma 4.1. In particular, we have the following homotopy fibration:

$$G_\omega \to \mathrm{Diff}_0(M) \to \mathcal{A}_{\Omega_\omega}. \tag{22}$$

Let $J \in \mathcal{A}_{\Omega_\omega}$. Suppose we perform the $b^+ = 1$ J−compatible inflation in Section 4.4 to ω to get a symplectic form ω' compatible with J. For ω' we have $\Omega_{\omega'}$, $\mathcal{A}_{\Omega_{\omega'}}$ and the homotopy fibration $G_{\omega'} \to \mathrm{Diff}_0(M) \to \mathcal{A}_{\Omega_{\omega'}}$. Moreover, if we pick $\tilde{J} \in \mathcal{A}_\omega$ and repeat the inflation to obtain a $\tilde{\omega}'$ compatible with $\tilde{J}$ and cohomologous to ω', then by Lemma 4.1 ω' and $\tilde{\omega}'$ have to be isotopic. We can prove properties analogous to Lemmas 5.1, 5.2 for $S_\omega, S_{\omega'}$, $\mathcal{A}_{\Omega_\omega}$, $\mathcal{A}_{\Omega_{\omega'}}$ with the same arguments. Similarly, we prove the analogue of Lemma 5.3 for $\pi_0(G_\omega), \pi_0(G_{\omega'})$, $\pi_i(Symp(M, \omega))$ and $\pi_i(Symp(M, \omega'))$ with $i \ge 1$ from the following commutative diagram:

$$\begin{tikzcd} G_\omega \arrow[r] \arrow[d] & \mathrm{Diff}_0(M) \arrow[r] \arrow[d] & \mathcal{A}_{\Omega_\omega} \arrow[d] \\ G_{\omega'} \arrow[r] & \mathrm{Diff}_0(M) \arrow[r] & \mathcal{A}_{\Omega_{\omega'}}. \end{tikzcd} \tag{23}$$

With essentially notational modifications we also prove the analogues of Theorem 5.4 and Theorem 1.3, except that $\pi_0(G_\omega)$ and $\pi_0(G_{\omega'})$ are in place of $\pi_0(Symp(M,\omega))$ and $\pi_0(Symp(M,\omega'))$ respectively.

To get equalities between $\pi_0(Symp(M,\omega))$ and $\pi_0(Symp(M,\omega'))$, we apply Lemma 2.2, Proposition 2.5 and the following lemma which identifies G_ω as the Torelli symplectic mapping class group for a rational surface.

Lemma 5.6. *Suppose (M,ω) is a symplectic rational surface. Then $G_\omega = Symp_h(M,\omega)$.*

Proof. Clearly, we have

$$Symp_0(M,\omega) \subset G_\omega = Symp(M,\omega) \cap \mathrm{Diff}_0(M) \subset Symp_h(M,\omega).$$

By [28] and [18], for a rational or ruled surface M, we have

$$Symp_h(M,\omega) \subset \mathrm{Diff}_0(M).$$

So we have $G_\omega = Symp_h(M,\omega)$ and $\pi_0(G_\omega) = Symp_h/Symp_0$ is just the Torelli symplectic mapping class group. ☐

The final remark is on the close relation between the topology of $Symp(M,\omega)$ and topology of the space of symplectically embedded balls. Recall Theorem 2.5 in [14] which reveals this relation:

Theorem 5.7 (Theorem 2.5 in [14])**.** *Let (M,ω) be a symplectic 4-manifold, and let c be a positive real number. Let $Emb_\omega(B^4(c),M)$ be the space of symplectic embeddings of the standard closed ball of capacity c, endowed with C^∞-topology. Suppose that*
(1) the space $Emb_\omega(B^4(c),M)$ is non-empty and connected, and
(2) the exceptional curve class E that one gets by blowing up an arbitrary ball $\iota_c \in Emb_\omega(B^4(c),M)$ cannot degenerate in $(\tilde{M},\omega_c)$, which is the blow-up of (M,ω) at the ball ι_c.
Then there is a homotopy fibration

$$Symp(\tilde{M},\omega_c) \to Symp(M,\omega) \to Emb_\omega(B^4(c),M).$$

Remark 5.8. *The condition (2) means that E has a unique embedded J−holomorphic representative for any ω_c-tamed almost complex structure J.*

Theorem 5.7 also works for a finite number of disjoint symplectic balls $\coprod B^4(c_i)$ if they satisfy the corresponding versions of conditions (1) and (2) in Theorem 5.7.

Applying the argument in [14] Theorem 1.6 and the above stability result of $Symp(M,\omega)$, we have the following stability result for the space of ball embeddings:

Lemma 5.9. *Notation as above, and let (M,ω) denote a rational surface with $\chi(M) \le 11$. Then the (weak) homotopy type of $Emb_\omega(B^4(c),M)$ is stable if ω_c does not cross any of the hyperplanes in the reduced cone defined by an element of $\mathcal{S}^{\le -2}$ the blowup $\tilde{M} = M\#\overline{\mathbb{C}P^2}$.*

This is to say that the group of symplectomorphisms $Symp(\tilde{M},\omega_c)$ is (weakly) homotopy equivalent to the group $Symp_p(M,\omega)$ fixing a point if c is smaller than the ω-area of any classes in $\mathcal{S}^{\le -1}$.

In [19], there is a conjectural higher homotopical generalization of Biran's ball packing stabilization theorem, which can be viewed as a level stability version of Lemma 5.9:

Conjecture 5.10. *Given any symplectic 4-manifold (M,ω), we consider the embedding of k disjoint balls of sizes $\{c_1\cdots,c_k\}$ respectively. We denote the space of such embeddings by $Emb_k(c_i)$. There is the following forgetful map from an element of $Emb_k(c_i)$ to their centers*

$$\sigma : Emb_k(c_i) \to Conf_k(M),$$

where $Conf_k(M)$ is the configuration space of ordered k points in M. Then for any $m \in \mathbb{Z}^+$, there exists an $\epsilon(m) > 0$ such that σ is m-connected when $c_i < \frac{\epsilon(m)}{k}$.

Theorem 1.1 and the multi-ball version Lemma 5.9 imply that this conjecture is true for $M = \mathbb{C}P^2$ with $k \le 8$. By [25] and [18], even for $k \ge 9$, $\pi_1(Symp(\mathbb{C}P^2\#k\overline{\mathbb{C}P^2},\omega))$ is still finitely generated and the (free) rank is bounded by $\frac{k(k+1)}{2}$, and hence we hope to establish in an upcoming work [17] the stability at π_1 level. Hence it is reasonable to speculate that a version of Theorem 1.3 still holds for $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k > 9$ (cf. Remark 5.5) and hence Conjecture 5.10 holds for $M = \mathbb{C}P^2$ with any number of embedded balls.

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