SYMPLECTIC ISOTOPY ON NON-MINIMAL RULED SURFACES

Abstract. We prove the stability of $Symp(X, w) \cap \mathrm{Diff}_0(X)$ for one-point blowup of irrational ruled surfaces and study their topological colimit. Non-trivial generators of $\pi_0[Symp(X, w) \cap \mathrm{Diff}_0(X)]$ that differ from Lagrangian Dehn twists are detected.

Contents

1. Introduction

In this note, we study some topological aspects of symplectomorphism groups, especially the symplectic isotopy problems of non-minimal irrational ruled surfaces.

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1.1. **Statements of the problems.** To set notations, let $M_g = \Sigma_g \times S^2$. By McDuff's classification results [16], any symplectic form is diffeomorphic to $\mu\sigma_{\Sigma_g} \oplus \sigma_{S^2}$ for some $\mu > 0$, up to diffeomorphism and normalization. Such classification result also holds in the blowups $M_g\#n\overline{\mathbb{C}P^2}$ [11]: if one picks up ω on $M_g\#n\overline{\mathbb{C}P^2}$, then after normalization ω has areas $(\mu, 1, e_1, \cdots, e_n)$ on the homology classes $B, F, E_1, \cdots, E_n$, where $\mu > 0, e_1 + e_2 < 1, 0 < e_i < 1$, $e_1 \geq e_2 \geq \cdots \geq e_n, e_1 < \mu$, choosing the standard basis $B, F, E_1, \cdots, E_n$ and associate coefficients $(\mu, 1, e_1, \cdots, e_n)$ to get a cohomology class, then the symplectic forms in this cohomology class are isotopic. cf. [15, 12]. After normalization, the vector $u = (\mu, 1, e_1, \cdots, e_n)$ determines all possible symplectic form cohomology classes and belongs to a convex region Δ^{n+1} in $\mathbb{R}^{n+1}$, whose boundary walls are n-dimensional convex regions given by linear equations. We will be concerned with symplectic deformations inside this region Δ^{n+1} for the n-points blowups.

In figure 1 below, we show the region corresponding to the one point blow up.

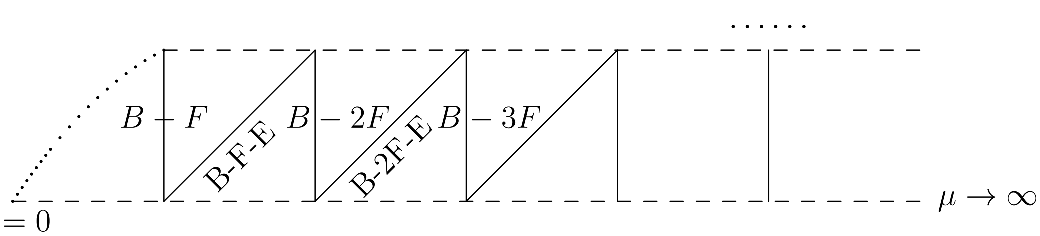

FIGURE 1. (Normalized) Symplectic cone of one-point blowup

Note that in Figure 1, the bottom dashed line means a symplectic minimal ruled surface which is the product $\Sigma_g \times S^2$ with $\omega(\Sigma_g)/\omega(S^2) = \mu$. The top line is symplectic minimal ruled surface which is the non-trivial bundle $\Sigma_g \tilde{\times} S^2$ with $\omega(\Sigma_g)/\omega(S^2) = \mu$. The interior of the cone is a one-point blowup of the minimal ruled surface, such that $\omega(\Sigma_g) = \mathcal{U}$, $\omega(S^2) = 1$, and $0 < \omega(E) = c < 1$. The very left chamber has round boundaries, which is a one-ball packing that's close to the volume constraint. Throughout this paper, we are going to assume $\mu \geq 1$, which means we'll never consider the chamber on the very left.

We can partition this cone Δ^{n+1} into countably many open-closed chambers (see Figure 2) by linear equations in $\mathbb{R}^{n+1}$ such that each chamber has the same arithmetically admissible cohomology classes, i.e. positive when evaluated on the corresponding homology classes. See section 2 for details.

The main concern of this paper is to address the topological stability of symplectomorphism group as the vector $u = (\mu, 1, e_1, \cdots e_n)$ changes within the chambers. Denote $G^n_{u,g}$ as the intersection of $Symp(M_g\#n\overline{\mathbb{C}P^2}, \omega)$ with $\mathrm{Diff}_0(M_g\#n\overline{\mathbb{C}P^2})$, the identity component of the diffeomorphism group. We will tackle the following conjecture on the topology of $G^n_{u,g}$:

Conjecture 1.1. *Let M be either M_g or one of its blowups, If two symplectic forms on M are represented by u_1 and u_2 belonging to the same chamber then $\pi_i(G^n_{u_1,g}) = \pi_i(G^n_{u_2,g}), \forall i \geq 0$.*

Variations of such conjecture can be addressed. For instance, one can prove stability for only a selective collection of these regions or only addressing the first n levels of homotopy groups.

This conjecture for minimal models has been established by McDuff [17] in for $g = 0$ and by Buse [4] for $g > 0$. The conjecture for the rational blow up cases has been proved by Anjos-Li-Li-Pinsonnault in [2].

In the current paper, we establish this conjecture for the one-point blowup cases. The many-point blowup cases will be studied in a future work [5].

Theorem 1.2. *The Conjecture 1.1 holds for $M_g\#\overline{\mathbb{C}P^2}$, $\forall g \geq 1$, with a symplectic form ω such that $[\omega] = [\mu, 1, c]$ and $\mu > g$, i.e. $\omega(\Sigma_g) > g\omega(S^2)$.*

More concretely, except for the first $2g-1$ chambers in Figure 1, on the following 2 types of chambers, stability holds.

The following Figure 2 represents the two types of stability chambers.

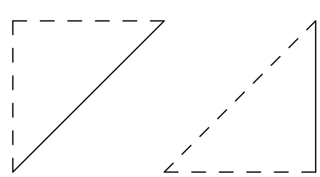

FIGURE 2. Stability chambers of one-point blowup

Once we establish the stability Theorem 1.2, this allows us to show the following topological colimit characterization.

Theorem 1.3. *For $M_g\#\overline{\mathbb{C}P^2}$, $\forall g\geq 1$, there exists a topological colimit $G^1_{\infty,g}$ for the groups $G^1_{u,g}$ as $\mu\to\infty$. Furthermore, $G^1_{u,g}$ has the homotopy type of the group $\mathcal{D}^1_g$, which is a diffeomorphisms group induced from the blowup of the fiberwise diffeomorphism of M_g endowed with the split complex structure, see section 5 and sequence 5 for details.*

A sufficient understanding of $\mathcal{D}^1_g$ from Theorem 5.7 allows us to conclude the following:

Corollary 1.4. *$G^1_{u,g}$ is disconnected for $g>1$ and $\mu>g$, where $u=[\mu,1,c_1]$.*

In [18] the following conjecture is proposed as open problem 14 for minimal ruled surfaces.

Conjecture 1.5 (Symplectic isotopy conjecture for ruled surfaces)**.** *For (M_g,ω), a symplectomorphism is symplectically isotopic to identity if and only if it is smoothly isotopic to identity.*

Our Corollary 1.4 shows that a corresponding conjecture is not true for the one-point blowup of ruled surfaces. There exist "exotic symplectomorphisms" that are smoothly but not symplectically isotopic to identity. Notice that for topological reasons, there are no Lagrangian spheres inside $M_g\#\overline{\mathbb{C}P^2}$, and hence those "exotic symplectomorphisms" are not generated by Dehn twists along Lagrangian spheres.

1.2. **Techniques of proofs.** The main difficulty in approaching the main conjecture is the absence of direct maps between the groups of symplectomorphism groups corresponding to two deformation equivalent symplectic forms on a given manifold.

McDuff's approach in [17] was to consider Kronheimer's fibration in [7]:

$$Symp(M,\omega)\cap \mathrm{Diff}_0(M)\to \mathrm{Diff}_0(M)\to \mathcal{T}_\omega, \tag{1}$$

where $\mathcal{T}_\omega$ represents the space of symplectic forms in the class $[\omega]$ and isotopic to a given form, and $\mathrm{Diff}_0(M)$ is the identity component of the diffeomorphism group. Moser's technique grants a transitive action of $\mathrm{Diff}_0(M)$ on $\mathcal{T}_\omega$ and hence gives us the fibration 1.

Note that there is no direct map between $Symp(M,\omega)$ when deforming ω. Following McDuff's work in [17][1], one uses the (weak) homotopy equivalence between $\mathcal{T}_\omega$ and the space $\mathcal{A}_\omega$ (which is the space of

[1] McDuff's original results were written in terms of homotopy fibration where the larger space of taming almost complex structures was used. Using the fact that the space of taming almost complex structures is homotopy equivalent to the one we

ω'-compatible almost complex structures, where ω' is any symplectic form isotopic to ω) to construct a homotopy fibration

$$Symp(M, \omega) \cap \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(M) \to \mathcal{A}_\omega. \tag{2}$$

By the inflation technique in section 4 one can relate (by direct inclusions of strata) the spaces $\mathcal{A}_\omega$'s for ω in different cohomology classes and hence prove stability results of $Symp(M, \omega)$. Thus at the heart of the matter remains establishing such strata and inclusions in the given setting. That involves:

(a) establishing the existence of sufficient inflation techniques

(b) existence of sufficient J holomorphic embedded (or nodal) curves for nongeneric almost complex structures so that the inflation can be performed.

Most techniques we are using here are concerning the study of s-paces of almost complex structures (not necessarily generic ones) and J-holomorphic curve they admit.

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2. The symplectic cone and its partition

By [12], if M is a closed, oriented 4-manifold with $b^+ = 1$, the symplectic canonical class is unique once we fix $u = [\omega]$, and we denote it by K_u.

Let $\mathcal{E}$ be the set of exceptional sphere classes, and $\mathcal{C}_M$ denote the symplectic cone.

In Theorem 4 of [12], Li-Liu showed that if M is a closed, oriented 4-manifold with $b^+ = 1$ and if the symplectic cone $\mathcal{C}_M$ is nonempty, then

$$\mathcal{C}_M = \{e \in P | 0 < |e \cdot E| \quad \text{for all} \quad E \in \mathcal{E}\}.$$

Note that the way we partition the normalized symplectic cone is by looking at the homology classes of potential symplectic curves. To that end, we will now fix some notation:

Definition 2.1. *Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic curves and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,*

$$K_\omega \cdot A = -A \cdot A - 2 + 2g(A). \tag{3}$$

For each $A \in \mathcal{S}_\omega$ we associate the integer

$$cod_A = 2(-A \cdot A - 1 + g).$$

use of compatible structures, for reasons explained in Section 3.4 we will use the compatible structure spaces

We can define $\mathcal{S}_u$, where $u = [\omega]$ accordingly, only using the cohomology data of ω. We are going to denote $\mathcal{S}_u^{<0}$ by the subset of $\mathcal{S}_u$ having negative self-intersections.

Remark 2.2.
- *By Lemma 2.4 in [2], $\mathcal{S}_u = \mathcal{S}_w$.*
- *By the main theorem in [13], the negative self-intersection classes that admit embedded representatives are exactly those that have positive pairings with the class ω. Namely, we can find some integral class u' that admits some embedded curve in those classes of $\mathcal{S}_u^{<0}$, and the symplectic inflation of [13] allows us to change the class u' into u.*

Note that the way we do partition for rules surfaces is by looking at the inequality $u \cdot A \leq 0$, where $A \in \mathcal{S}_u^{<0}$ and $Cod(A) > 0$. In particular, by **wall**, we refer to those subsets $\{u | u \cdot A = 0, A \in \mathcal{S}_u^{<0}$ of the symplectic cone.

Hence when $M = M_g \# \overline{\mathbb{C}P^2}$, Figure 1 represents $\mathcal{C}_M$, when normalizing $\omega(F)$ to be 1.

Explicitly, the chambers we obtain in the one-point blow-up cases are given by the following inequalities:

- The $2k+1$-th region is given my $u \cdot [B - kF - E] > 0$,
  $u \cdot [B - (k+1)F] \leq 0$.
- The $2k$-th region is given by: $u \cdot [B - kF] > 0$,
  $u \cdot [B - kF - E] \leq 0$.
- Top: $u \cdot E < 1$; Bottom: $u \cdot E > 0$.

3. Homotopy fibration and the stratification of $\mathcal{A}_u$

The homotopy fibration (2) presented in the introduction has been fruitfully applied to study the topology of G_ω, especially in dimension 4, cf. [17], [4], [1] etc. Now let's focus on the case of non-minimal ruled surfaces. Following the same strategies employed by [8], one needs to find a sufficiently fine stratification of the spaces of almost complex structures and show that they only differ by the addition of such (finite codimension) strata when ω is crossing the walls of the chambers of the arithmetic regions in Δ^{n+1}.

Note that in the non-minimal case, cohomologous forms are not known to be isotopic, in contrast to the minimal case. We'll consider a larger space $\mathcal{A}_u$, which is the space of almost complex structures that are compatible with some ω s.t. $[\omega]=u$. A discussion on this topic will be included in section 3.3.

3.1. **Proposed stratification of the spaces of almost complex structures.** First we define the prime subset of the space of almost complex structures $\mathcal{A}_{u,\mathcal{C}}$ labeled by set $\mathcal{C}\subset\mathcal{S}_u^{<0}$ for a given isotopy class of ω as following:

Definition 3.1. *A subset $\mathcal{C}\subset\mathcal{S}_u^{<0}$ is called admissible if*

$$\mathcal{C}=\{A_1,\cdots,A_i,\cdots,A_q|\quad A_i\cdot A_j\geq 0,\quad \forall i\neq j\}.$$

Given an admissible subset $\mathcal{C}$, we define the real codimension of the label set $\mathcal{C}$ as

$$cod(\mathcal{C})=\sum_{A_i\in\mathcal{C}} cod_{A_i}=\sum_{A_i\in\mathcal{C}}2(-A_i\cdot A_i-1+g_i).$$

Define the **prime subset**

$\mathcal{A}_{u,\mathcal{C}}:=\{J\in\mathcal{A}_u|A\in\mathcal{S}_u^{<0}$ *has an embedded J-hol representative if and only if* $A\in\mathcal{C}\}$.

And if $\mathcal{C}=\{A\}$ contains only one class A, we will use $\mathcal{A}_A$ for $\mathcal{A}_{\{A\}}$.

Notice that these prime subsets are disjoint and we have the decomposition $\mathcal{A}_u=\amalg_{\mathcal{C}}\mathcal{A}_{u,\mathcal{C}}$.

We define a filtration according to the codimension of these prime subsets:

$$\cdots\subset\mathcal{X}_{u,2n+1}\subset\mathcal{X}_{u,2n}(=\mathcal{X}_{u,2n-1})\subset\mathcal{X}_{u,2n-2}\ldots\subset\mathcal{X}_{u,2}(=\mathcal{X}_{u,1})\subset\mathcal{X}_{u,0}=\mathcal{A}_u,$$

where $\mathcal{X}_{u,j}:=\amalg_{\mathrm{cod}(\mathcal{C})\leq j}\mathcal{A}_{u,\mathcal{C}}$ is the union of all prime subsets having codimension no less than j.

We define the open strata to be the complement of positive codimension strata in $\mathcal{A}_u$. Namely, we denote $\mathcal{A}_{u,open}$ as $\mathcal{X}_{u,0}-\mathcal{X}_{u,2}$.

Remark 3.2. *Note that we don't have control of what type of classes are J-holomorphic embedded in $\mathcal{A}_{u,open}$. For different $J\in\mathcal{A}_{u,open}$, there could be different embedded curves in the section classes $B, B+F, B+2F,\cdots$; see proof of Lemma 4.3 for details.*

For $\mathcal{S}_u^{<0}$, the following shows that the prime subsets are well behaved analytically.

Proposition 3.3. *Let (X,ω) be a symplectic 4-manifold. Suppose $U_{\mathcal{C}}\subset\mathcal{A}_\omega$ is a subset characterized by the existence of a configuration of embedded J-holomorphic curves $C_1\cup C_2\cup\cdots\cup C_N$ of positive* *codimension as in Definition 2.1 with* $\{[C_1], [C_2], \cdots, [C_N]\} = \mathcal{C}$*. Then* $U_{\mathcal{C}}$ *is a co-oriented Fréchet suborbifold of* $\mathcal{A}_\omega$ *of (real) codimension* $2N - 2c_1([C_1] + \cdots + [C_N]) = \sum_i K \cdot [C_i] - [C_i]^2$.

In particular, this statement covers the case where (X, ω) *is a ruled surface and* C_i'*s all have negative squares.*

Proof. Firstly, it suffices to show the above result in the Banach setting. In particular, Theorem 2.1.2 in [6] showed that the space of the nodal curve in those fixed classes is a finite co-dimensional Banach analytic subset. Then use the argument in Appendix B of [1], one can construct a local chart with codimension $2N - 2c_1([C_1] + \cdots + [C_N])$ of $\mathcal{A}_u$ at each point of the space of J so that there is an embedded curve in each component.

The orbifold structure comes from Teichmller space to the moduli space of Riemann surfaces of genus g. Hence the chart is quotient by at most a finite group. Note that this follows Lemma 2.6 in [17]. □

Corollary 3.4. *The prime subsets* $\mathcal{A}_{\omega,\mathcal{C}}$ *are suborbifolds of* $\mathcal{A}_\omega$ *codimension of* $Cod(\mathcal{A}_{\omega,\mathcal{C}})$*.*

Proof. $\mathcal{A}_{\omega,\mathcal{C}}$ is a subset of $U_{\mathcal{C}}$, and the complement of $\mathcal{A}_{\omega,\mathcal{C}}$ in $U_{\mathcal{C}}$ is a union of $U_{\mathcal{C}_i}$, where $\mathcal{C}_i$'s are admissible sets containing $\mathcal{C}$. Hence $\mathcal{A}_{\omega,\mathcal{C}}$ is a suborbifolds of the same codimension as $U_{\mathcal{C}}$ in $\mathcal{A}_\omega$. □

3.2. **Inflation as the means of transportation between the spaces of almost complex structures.** McDuff [17] and Buse [4] used version of symplectic inflation keeping track of an almost complex structure J to study the structure of the spaces of almost complex structures. As discussed in [2] Section 4.4, their proofs made the unwarranted assumption that for every ω-tame J and every J-holomorphic curve C one can find a family of normal planes that is both J invariant and ω-orthogonal to TC. This is true only if ω is compatible with J at every point of C. A correct version of their statements, which works with the proof provided in [17] for positive self intersection curves and in Theorem 1.1 in [4] for negative self-intersection curves, combines as:

Theorem 3.5. *For a 4-manifold* M*, given a compatible pair* (J, ω)*, one can inflate along a* J*-holomorphic curve* Z*, so that there exist a symplectic form* ω' *taming* J *such that* $[\omega'] = [\omega] + tPD(Z), t \in [0, \mu)$ *where* $\mu = \infty$ *if* $Z \cdot Z \geq 0$ *and* $\mu = \frac{\omega(Z)}{(-Z \cdot Z)}$ *if* $Z \cdot Z < 0$*.*

As further explained in [2], for four-dimensional manifolds M with $b^+ = 1$, (such as the blow-up ruled surfaces) the strategy is to perform the above inflation to obtain a J-tame ω' in the correct cohomology class. Then the result of Li and Zhang [14] giving the comparison of tame/compatible cones, one has an ω' compatible with the given J. Anjos et all call this $b^+ = 1$ $J-$**compatible inflation**.

3.3. **Keeping track of isotopy classes during the $b^+ = 1$ $J-$compatible inflation.** As previously mentioned, cohomologous forms are not known to be isotopic in the cases of blow-ups, in contrast to the minimal case.

When inflating compatible with J in the larger space $\mathcal{A}_u$ of almost complex structures compatible with some symplectic form in the cohomology class u, we must exhibit the following properties:

• **All the connected components of both $\mathcal{A}_u$ and $\mathcal{T}_u$ are homotopic to each other and there is a canonical bijection between the two sets of components**. By the argument in Lemma 4.1 in [2], $\mathcal{A}_\omega$ is a path connected component of $\mathcal{A}_u$ and is canonically homotopy equivalent to $\mathcal{T}_\omega$.

In fact, $\mathcal{T}_\omega$ and $\mathcal{A}_\omega$ correspond to each other under the canonical bijection between the sets of path connected components of $\mathcal{T}_u$ and $\mathcal{A}_u$ in Lemma 4.1 in [2].

• **Performing inflations from one cohomology class to another preserves the isotopy components of the spaces $\mathcal{A}_u$ and $\mathcal{T}_u$.** Let $J \in \mathcal{A}_\omega$ such that $[\omega] = u$. Suppose that we perform the $b^+ = 1$ $J-$compatible inflation in Section 3.2 to ω to first get a symplectic form ω' taming J. Then by the cone results of Li-Zhang in [14], one can concludes that J is in fact compatible with some symplectic form ω'' in the same cohomology class u' of ω'.

Claim 3.6. *i) For another $\tilde{J} \in \mathcal{A}_\omega$, after performing the two steps in the $b^+ = 1$ $\tilde{J}-$compatible inflation we obtain a symplectic form $\tilde{\omega}''$ in the same isotopy component of $\mathcal{T}_{u'}$ as ω''.*

ii) If we perform the opposite direction two step $b^+ = 1$ *inflation for any other $J'' \in \mathcal{A}_{\omega''}$ then we obtain an $\omega''' \in \mathcal{T}_\omega$.*

From this Claim, we conclude that the inflation process does not change connected components.

Proof. First, let's choose the connected component explicitly. The proof for the Claim i) and ii) for any other components follow from the first bullet in this section.

Let J_{split} be the product complex structure on $\Sigma_g \times S^2$. Let J_{std} be **the blowup of J_{split}, at a point p of a fixed fiber which is not the intersection point on the base.**

As we'll see blow, for any cohomology class u, there is a preferred connected component (denoted $\mathcal{A}_\omega$) of $\mathcal{A}_u$ to be the one that contains the J_{std}, and choose all other $\mathcal{A}_{\omega'}$ or $\mathcal{T}_{\omega'}$ accordingly.

Namely, for any u, there exist an isotopy class $\bar{\omega}$ in u so that J_{std} is compatible with some form in the isotopy class, namely, $J \in \mathcal{A}_{\bar{\omega}}$. The reason is we can always start from a Kähler ruled surface and inflate along the embedded fiber class curve to achieve any cohomology class in figure 2. Hence the standard J_{std} tames some form in every class. Then the comparison of the tame and compatible cone by Li-Zhang affirms that J_{std} is compatible with some form in every class. Notice that this gives the canonical choice of the connected component of $\mathcal{A}_u$ for any u. And this proves fact 1.

Then we prove Claim ii), we use the canonical choice of components given by J_{std}:

Recall that both J_{std} and J'' blongs to the same path connected component $\mathcal{A}_{\omega''}$.

We then define the product space $P_{\omega''} = \{(\omega, J) | \mathcal{T}_u \times \mathcal{A}_{\omega''} : \omega \text{ is compatible with } J, [\omega] = u\}$. Consider the projection from $P_{\omega''}$ to $\mathcal{A}_{\omega''}$.

Notice that the projection onto the J factor and ω factor both have convex and hence contractible fibers, by [18] section 3.5. And the space $\mathcal{A}_{\omega''}$ is connected. Then the product space $P_{w''}$ is also path connected. Hence we know that the form ω''' described in the statement of Claim ii) lives in the same path-connected component as the original ω.

□

Now we always have a well defined connected component $\mathcal{T}_{u'}$ and the corresponding space $\mathcal{A}_{u'}$, which is the space of J compatible with some form in $\mathcal{T}_{u'}$ that contains J_{std}. For ω' we have $\mathcal{T}_{\omega'}$, $\mathcal{A}_{\omega'}$ and the homotopy fibration $G_{\omega'} \to \mathrm{Diff}_0(M) \to \mathcal{A}_{\omega'}$.

Similarly, we are going to use the following diagram in the proof of 1.2

$$
\begin{tikzcd}
G_\omega \arrow[r] \arrow[d] & \mathrm{Diff}_0(M) \arrow[r] \arrow[d] & \mathcal{A}_\omega \arrow[d] \\
G_{\omega'} \arrow[r] & \mathrm{Diff}_0(M) \arrow[r] & \mathcal{A}_{\omega'}.
\end{tikzcd}
\tag{4}
$$

3.4. **A historical detour to the minimal cases.** We find it informative to explain the proof strategy from McDuff [17] and Buse [4] in the minimal cases, to show how we can evolve our strategy in the non-minimal cases.

When the symplectic manifold is a minimal ruled surface the symplectic cone is given by a line and the finite codimension strata are given by curves in classes $B-kF$; thus they are labeled $\mathcal{A}^k_{g,\mu}$ and $\mathcal{A}^{open}_{g,\mu}$ respectively. McDuff showed that for each J *there is a foliation of* M_g *with leaves embedded* J *holomorphic curves* F and used this to show that right inclusion $\mathcal{A}_{g,\mu} \subset \mathcal{A}_{g,\mu+\epsilon}$ for all $\mu, \epsilon > 0, g > 0$ or for all $\mu > 1, \epsilon > 0, g = 0$.

In her work, this right inclusion is done regardless of strata; however, left inclusions ought to be proved stratum by stratum (including the open stratum) using inflation methods along embedded curves with positive self-intersections.

More specifically, she used curves in base class B (with a notable restriction on $\mu > g$ for the higher genus cases) to prove the left inclusion $\mathcal{A}^{open}_{g,\mu} \supset \mathcal{A}^{open}_{g,\mu+\epsilon}$.

The existence of sufficient embedded positive curves in the strata $\mathcal{A}^k_{g,\mu}$ proved difficult in McDuff's work in the cases $g > 0$, although possible in the rational cases; later, Buse developed the inflation technique along negative curves to complete the left inclusions for the strata with finite codimension and complete the inclusions $\mathcal{A}^k_{\mu} \supset \mathcal{A}^k_{\mu+\epsilon}$ for all $k > 1$.

As a byproduct of these techniques and the homotopy fibration (2) they verified the Conjecture 1.1 holds for the following cases.

Theorem 3.7.
- *McDuff [17] when $g = 0$, the homotopy type of the groups G^0_μ is unchanged on all intervals $(n, n+1], n$ for any nonzero natural number n.*
- *Buse [4] when $g > 0$, the homotopy type of the groups G^g_μ is unchanged on all intervals $(n, n+1]$ for any nonzero natural number $n \geq \lfloor g/2 \rfloor$.*

Moreover, this stability results allowed McDuff to show that a homotopy colimit $G_{\infty,g}$ exists and she shows that this group is homotopy equivalent to a smooth model group, namely the group $\mathcal{D}^0_g$ of smooth fiberwise diffeomorphisms of M_g. Among the topological consequences of this result we note that $\mathcal{D}^0_g$,$G_{\infty,g}$ and consequently $G_{g,\mu}$ *are connected.*

4. Stability of strata of $\mathcal{A}_\omega$ in the one point blowup cases

4.1. **Our strategy.** For the treatment of nonminimal cases involving a multiple blow up, one approaches a similar strategy. However, the structure of the chambers is more complicated and one has to deal with several additional problems.

Notation convention: whenever unnecessary, we are going to omit the genus subscript in our spaces and n (the blowup number, which always equals 1 here), for example, we'll write $\mathcal{A}_\omega$ instead of $\mathcal{A}^n_{g,\omega}$.

Firstly, we use a singular foliation result of Zhang [**Zhang16**] to proceed with the right inflation of the total(unstratified) spaces of compatible almost complex structures. Then one has to produce additional curves to deal with inflation *within the chambers*. Also, notably, a similar restriction on the symplectic class with regards to the g appears (due to constraints arising from Gromov invariants computations) when dealing with leftwards inflation on the open stratum. The general cases for several points blow-ups will be treated in future work [5];

The present paper will only treat the one point blow up cases for any $g > 0$; in this instance, the necessary foliation result of Zhang [**Zhang16**] (see also Shevshishin-Simirov [**SS17elliptic**]) translates into the following:

Lemma 4.1. *Let (Z, ω) be a symplectic irrational ruled 4-manifold diffeomorphic to $S^2 \times \Sigma_g \# \overline{\mathbb{C}P}^2$, and let J be an ω-compatible almost-complex structure. Then Z admits a* **singular foliation** *given by a proper projection $\pi : Z \to Y$ where Y is a smooth compact surface of genus g such that*

i) there is a singular value $y^* \in Y$ such that π is a foliation over the leaf space $Y - y^*$, with the fiber $\pi^{-1}(y)$, $y \in Y - y^*$, represented by an embedded J-holomorphic rational curve in the class F;

ii) the fiber $\pi^{-1}(y^*)$ consists of the two exceptional J-holomorphic smooth rational curves in the classes $F - E$ and E.

This will be used with both the right inflation as well as the chamber positioning which in these cases involves increasing or decreasing the blow-up sizes (vertical inflations). Our Proposition 4.2 and the table 1 in section 4 explain the strategy.

We are also able to show the existence of a homotopy colimit and provide a smooth diffeomorphism model in Section 5. However the topology of this group is more difficult to study so we content ourselves to show that, in contrast to the minimal case, both this smooth model and consequently the symplectomorphism groups *are not connected*.

4.2. **Inflating.** Throughout this section, We are going to denote the following $\mathcal{A}_u$ or $\mathcal{A}_{u,\mathcal{C}}$ by the canonical component.

This section is concerned with proving the required stability (invariance) of strata in the spaces of almost complex structures $\mathcal{A}_\omega$ when we consider the one point blow-ups for all $g > 0$. The table reflects our strategy of proof in Theorem 1.2.

The general idea is that moving right (toward infinity) is always easy, but moving left is harder. And we start with a point on the left, name a target in a chamber and do large scale or small scale moves to hit the target.

Proposition 4.2. *In the following cases, the strata have inclusion relations:*

(1) $\mathcal{A}_{u_1,\mathcal{C}} = \mathcal{A}_{u_2,\mathcal{C}}$, *if* $u_1 = [\mu, 1, c_1], u_2 = [\mu, 1, c_2]$, $\forall \mathcal{C} \subset S^{<0}$.

(1') *For the open stratum,* $\mathcal{A}_{u_1,open} = \mathcal{A}_{u_2,open}$, *if* $u_1 = [\mu, 1, c_1], u_2 = [\mu, 1, c_2]$, *and* $\mu > g$.

(2) *For any stratum, including the open stratum,* $\mathcal{A}_{u,\mathcal{C}} \subset \mathcal{A}_{u',\mathcal{C}}$, *if* $u = [\mu, 1, c], u' = [\mu+\epsilon, 1, c]$, $\forall \mathcal{C} \subset S^{<0}$ *and for all* $\mu > 1, \epsilon > 0$.

(3) $\mathcal{A}_{u,open} \supset \mathcal{A}_{u',open}$, *where* $u = [\mu, 1, c], u' = [\mu + \epsilon, 1, c]$, *and for all* $\mu > g, \epsilon > 0$.

(4) $\mathcal{A}_{u,\mathcal{C}} \supset \mathcal{A}_{u',\mathcal{C}}$, $u = [\mu, 1, c], u' = [\mu + \epsilon, 1, c]$, $\forall \emptyset \neq \mathcal{C} \subset S^{<0}$ *and for all* $\mu > 1, \epsilon > 0$.

In conclusion, any two forms in the same chamber has the same strata in every level, the strata of $\mathcal{A}_u$ *and* $\mathcal{A}_{u'}$ *that are labeled by the same curve are the same, if* $\mu > g$.

Proof. (2) is Lemma 4.5, (3) is Lemma 4.6 (4) is Lemma 4.7.

(1) and (1') follows from Lemma 4.4. □

To prove Proposition 4.2, we first need to show the existence of embedded J holomorphic curves. The following proposition establishes all the curve existence results(some results here repeat Zhang's foliation result)

Proposition 4.3. *Compendium of J-holomorphic curves on* $S^2 \times \Sigma_g \# \overline{\mathbb{C}P}^2$.

(1) *For any* $J \in \mathcal{A}_u$, *there are embedded J-holomorphic curves in the classes* F, $F - E$ *and* E *by [***Zhang16***].*

(2) *For any J in a positive co-dimensional stratum* $\mathcal{A}_{u,C}$, *where* C *is either* $B - kF$ *or* $B - kF - E$, C *is represented by an embedded J-holomorphic curve by the definition of the stratum.*

(3) *For* $\mathcal{A}_{u,open}$, *there is an embedded J-holomorphic curve in some* $B + xF$, *where* $x \leq g$.

Proof. (1) Theorem 1.6 in [**Zhang16**] gives the singular foliation where the smooth leaves gives the embedded F, and the singular leaf gives the embedded $F - E$.

(2) It follows by definition.

Direction	Strata	Class to inflate	Proof	Size/Note
$\uparrow$ or $\downarrow$	$\mathcal{A}_{u,\mathcal{C}}$	E, $F-E$,F	Lemma 4.4	Foliation and exceptional curve
$\uparrow$ or $\downarrow$	$\mathcal{A}_{u,open}$	$B+xF$,$F-E$	Lemma 4.4	Foliation and exceptional curve
$\longrightarrow$	Any strata	F	Lemma 4.5	Foliation allows any size
$\longleftarrow$	$\mathcal{A}_{u,open}$	$B+xF$, $x \leq g$	Lemma 4.6	left to the chamber along $B+xF$, then Lemma 4.4
$\longleftarrow$	$\mathcal{A}_{B-kF-E}$	$B-kF-E$	Lemma 4.7 1st bullet	left to the chamber along $B-kF-E$, then Lemma 4.4
$\longleftarrow$	$\mathcal{A}_{B-kF}$	$B-kF$	Lemma 4.7 2nd bullet	left to the chamber along $B-kF$, Lemma 4.4.

TABLE 1. Inflation process

(3) It was proved by Li-Liu [12] that, if $M = \Sigma_g \times S^2$ or its blowup, where $g > 0$ and $C = pB + qF$, then

$$\mathrm{Gr}(C) = (p+1)^g, \quad \text{provided that } \ k(C) \geq 0.$$

Here $k(C)$ means the virtual dimension of the moduli space of curves in class C.

In particular, $\mathrm{Gr}(C) \neq 0$ provided that $q \geq g-1$. When $g = 0$, $\mathrm{Gr}(C) = 1$ for all classes C with $p, q \geq 0$ and $p+q > 0$.

In general, for the genus g cases we always have $Gr(B+gF) = 2^g > 0$, and expect to have $k(B+gF) = \frac{1}{2}([2B-2F-\cdots]\cdot[B+gF]+[B+gF]^2) = 2g+2-2g \geq 0$.

Now we have a stable curve in the class $B+gF$ for any J. Since $J \in \mathcal{A}_{u,open}$, there are no curves with negative self-intersection, meaning that for all curves, both B and F have non-negative coefficients. Now, looking at the stable representative, we know there is exactly one component where B has coefficient 1, due to the non-negativeness of B-coefficient. And that component has to be $B+xF$ for some $x \leq g$, due to the non-negativeness of F-coefficient.

□

Lemma 4.4. *$\mathcal{A}_{u_1,\mathcal{C}} = \mathcal{A}_{u_2,\mathcal{C}}$, if $u_1 = [\mu, 1, c_1], u_2 = [\mu, 1, c_2]$, $\forall \mathcal{C} \subset S^{<0}$. Moreover, for the open stratum, $\mathcal{A}_{u_1,open} = \mathcal{A}_{u_2,open}$, if $u_1 = [\mu, 1, c_1], u_2 = [\mu, 1, c_2]$, and $\mu > g$.*

Proof. Let's assume $c_2 > c_1$.

We'll prove $\mathcal{A}_{u_1,\mathcal{C}} \subset \mathcal{A}_{u_2,\mathcal{C}}$, and $\mathcal{A}_{u_1,\mathcal{C}} \supset \mathcal{A}_{u_2,\mathcal{C}}$ (including the open strata).

$\mathcal{A}_{u_1,\mathcal{C}} \supset \mathcal{A}_{u_2,\mathcal{C}}$ is always easy since we always have an embedded E, which we can inflate along.

Now for $\mathcal{A}_{u_1,\mathcal{C}} \subset \mathcal{A}_{u_2,\mathcal{C}}$, Proposition 4.3 grants the following curves that we can inflate along:

(1) For the open stratum: We inflate along $B+xF$ and $F-E$, and we want

$$[\omega_{t_1,t_2}] = [\mu,1,c_1] + t_1[x,1,0] + t_2[1,0,1] = [\mu+t_1+t_2x, 1+t_2, c_1+t_2],$$

such that $\mu+t_1+t_2x = \mu(1+t_2)$ and $c_1+t_2 = c_2(1+t_2)$.

Solve this system of two linear equations, we have $t_1 = (\mu - x)t_2$, and $(1-c_2)t_2 = c_2 - c_1$. Hence we have a solution such that both t_1 and t_2 are positive.

(2) For $\mathcal{A}_{B-kF}$: We inflate along $B-kF$ and $F-E$, and we want

$$[\omega_{t_1,t_2}] = [\mu,1,c_1] + t_1[-k,1,0] + t_2[1,0,1] = [\mu+t_1-kt_2, 1+t_2, c_1+t_2],$$

such that $\mu+t_1+kt_2 = \mu(1+t_2)$ and $c_1+t_1+t_2 = c_2(1+t_2)$.

Solve this system of two linear equations, we have $t_1 = (\mu + k)t_2$, and $+(1-c_2)t_2 = c_2 - c_1$. Hence we have a solution such that both t_1 and t_2 are positive.

(3) For $\mathcal{A}_{B-kF-E}$: We inflate along $B-kF-E$ and $F-E$, and we want

$$[\omega_{t_1,t_2}] = [\mu,1,c_1] + t_1[-k,1,1] + t_2[1,0,1] = [\mu+t_1-kt_2, 1+t_2, c_1+t_1+t_2],$$

such that $\mu+t_1+kt_2 = \mu(1+t_2)$ and $c_1+t_1+t_2 = c_2(1+t_2)$.

Solve this system of two linear equations, we have $t_1 = (\mu + k)t_2$, and $t_1 + (1-c_2)t_2 = c_2 - c_1$. Hence we have a solution such that both t_1 and t_2 are positive.

$\square$

Lemma 4.5. *For any stratum, including the open strata, $\mathcal{A}_{u,\mathcal{C}} \subset \mathcal{A}_{u',\mathcal{C}}$, $u = [\mu,1,c], u' = [\mu+\epsilon,1,c]$, and for all $\mu > 1, \epsilon > 0$.*

Proof. By [**Zhang16**] Theorem 1.6, we known that for each $J \in \mathcal{A}_{u,\mathcal{C}}$, through each point of M there is a stable J-holomorphic spheres representing the fiber class $F = [pt \times S^2]$.

Then we can inflate along the embedded curve F. And let's start with $u = [\mu,1,c]$.

By inflating, we obtain a form in $tP.D[F] + [\mu,1,c]$= $[\mu+t,1,c]$, $\forall t \in [0,\infty)$.

Hence the proof.

$\square$

Lemma 4.6. *For $\mathcal{A}_{u,open} \supset \mathcal{A}_{u',open}$, where $u = [\mu, 1, c], u' = [\mu+\epsilon, 1, c]$, and for all $\mu > g, \epsilon > 0$.*

Proof. By the proposition 4.3, we have an embedded curve in the class $B + xF$ for some $x \leq g$.

Then we can inflate along it. And let's start with $u = [\mu, 1, c]$.

By inflating, we obtain a form in $tP.D[B+xF]+[\mu, 1, c]= t[x, 1, 0]+ [\mu + t, 1, c]$, which normalized to

$$\left(\frac{tx + \mu}{1+t}, 1, \frac{c}{1+t}\right),$$

$\forall t \in [0, \infty)$.

Note that $\lim_{t\to\infty} \frac{tx + \mu}{1+t} = x \leq g$, which covers all the $\mu > g$. □

Lemma 4.7. $\mathcal{A}_{u,\mathcal{C}} \supset \mathcal{A}_{u',\mathcal{C}}$*, $u = [\mu, 1, c], u' = [\mu+\epsilon, 1, c]$, $\forall \emptyset \neq \mathcal{C} \subset S^{<0}$ and for all $\mu > 1, \epsilon > 0$.*

Proof. Now let's deal with the inflation when the area of the base is getting smaller.

- Pick a $J \in \mathcal{A}_{B-kF-E}$, we want the inflation process to reach as far left as the chamber labeled in 1 by the curve $B - kF - E$, then we inflate along this curve as follows.

  Let's assume that we start with $[\omega] = [\mu, 1, c_1]$ and we want a J-tame $\omega' \in [\omega'] = [\mu', 1, c_1']$ where $1 < \mu' < \mu$. Then we inflate along $B - kF - E$ and there is a family $\omega_t, s.t.[\omega_t] = [\mu + t, 1 + t, c_1 + t]$. And we want $\frac{\mu+t}{1+t} = \mu'$, and this means that $t = \frac{\mu-\mu'}{\mu'-1}$.

  Note that the range of t is given by
  $\mu + t - k(1 + t) - (c_1 + t) > 0$, i.e. $\mu - c_1 > kt$ or $0 \leq t < \frac{\mu-c_1}{k}$
- Pick a $J \in \mathcal{A}_{B-kF}$, we want the inflation process to reach as far left as the chamber labeled in 1 by the curve $B - kF$, then we inflate along this curve as follows.

  Let's assume that we start with $[\omega] = [\mu, 1, c_1]$ and we want a J-tame $\omega' \in [\omega'] = [\mu', 1, c_1']$ where $1 < \mu' < \mu$. Then we inflate along $B - kF$ and there is a family $\omega_t, s.t.[\omega_t] = [\mu + t, 1 + t, c_1]$. And we want $\frac{\mu+t}{1+t} = \mu'$, and this means that $t = \frac{\mu-\mu'}{\mu'-1}$.

  Note that the range of t is given by
  $\mu + t - k(1 + t) - (c_1) > 0$, i.e. $\mu - c_1 > (k-1)t$ or $0 \leq t < \frac{\mu-c_1}{k-1}$

Note that after inflation in either of the bullets above, we move slant left as far as we desire. And if we reach some point in the chamber as in Figure 1, then to reach any point in the same chamber, we use Lemma 4.4. This grants that we can reach any point horizontally as we claimed in the Lemma.

□

In conclusion, assuming $\mu, \mu' > g$, the overlapping strata of $\mathcal{A}_u$ and $\mathcal{A}_{u'}$ are the same.

4.3. **Proof of Theorem 1.2.** The Proposition 4.8 following McDuff Corollary 2.3 in [17] allows us to show the stability of the symplectomorphism group:

Proposition 4.8. *For any* $u = [\mu, 1, c_1], u' = [\mu + \epsilon, 1, c_2]$, *and* $u'' = [\mu+\epsilon+\epsilon', 1, c_2], \mu > g, \epsilon, \epsilon > 0$ *there are maps* $\mathcal{A}_u \to \mathcal{A}_{u'}$ *and* $G_u \to G_{u'}$ *that are well defined up to homotopy and make the following diagrams homotopy commute:*

$$(a) \qquad \begin{array}{ccccc} G_u & \to & \mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) & \to & \mathcal{A}_u \\ \downarrow & & \downarrow = & & \downarrow \\ G_{u'} & \to & \mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) & \to & \mathcal{A}_{u'}, \end{array}$$

$$(b) \qquad \begin{array}{ccc} G_u & \to & G_{u'} \\ & \searrow & \downarrow \\ & & G_{u''} \end{array} \quad .$$

Proof. The maps $\mathcal{A}_u \to \mathcal{A}_{u'}$ are the inclusions $\mathcal{A}_u \subset \mathcal{A}_{u'}$. Since G_u is the fiber of the map $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{A}_u$, there are induced maps $G_u \to G_{u'}$ making diagram (a) homotopy commute. The rest is obvious.

□

The main Theorem 1.2 immediately follows from Proposition 4.2 along with the above Proposition 4.8.

5. Singular foliations and topological colimit

The stability Theorem 1.2 grants us that the homotopy colimit $G^1_{\infty,g}$ (for each horizontal line fixing the blowup size) exists.

We are going to use the relationship between the space of singular foliations and the space of almost complex structures to establish a smooth diffeomorphism model for $G^1_{\infty,g}$. We will show that this smooth diffeomorphism model is disconnected and hence conclude that $G^1_{\infty,g}$ is disconnected.

Proposition 4.8 shows that the homotopy colimit exists.

Now we prepare and give the proof of Theorem 5.7.

Since $\mathcal{A}_u$ is an open subset of $\mathcal{A}_{u'}$ where $u' = [\mu', 1, c]$, s.t. $\mu' = \mu + \epsilon$ for all $\epsilon > 0$, the homotopy colimit $\lim_\mu \mathcal{A}_u$ of the spaces $\mathcal{A}_u$ is homotopy equivalent to the union $\mathcal{A}_\infty = \cup_\mu \mathcal{A}_u$.

Recall J_{split} be the product complex structure on $\Sigma_g \times S^2$. Let J_{std} be **the blowup of J_{split}, at a point p of a fixed fiber which is not the intersection point on the base.**

Lemma 5.1. *There is a map* $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{A}_\infty$ *which induces a homotopy fibration, having* $G^1_{g,\infty}$ *as the homotopy fiber.*

Proof. Because J_{std} is compatible with some $\omega \in \mathcal{T}_u$ the map $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{T}_\mu$ lifts to

$$\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to (\mathcal{T}_u, \mathcal{A}_u): \quad \phi \mapsto (\phi_*(\omega_\mu), \phi_*(J_{std})).$$

Composing with the projection to $\mathcal{A}_u$ we get a map

$$\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{A}_u: \quad \phi \mapsto \phi_*(J_{std})$$

that is not a fibration but has homotopy fiber G_u.

Then by Proposition 4.8 (b), we are able to construct an action $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{A}_\infty$ which is compatible with all such actions $\mathrm{Diff}_0(M_g \# \overline{\mathbb{C}P^2}) \to \mathcal{A}_u$.

□

To understand $\mathcal{A}_\infty$, let us first introduce a space Fol of singular foliations of $\Sigma_g \times S^2 \# \overline{\mathbb{C}P^2}$ as in Definition 5.2. In particular, this space only contains the foliation with one nodal fiber defined as follows:

Definition 5.2. *A* **singular foliation** *by* S^2 *of* $\Sigma_g \times S^2 \# \overline{\mathbb{C}P^2}$ *is defined as a foliation with smooth embedded spherical leaves in the* $F = [pt \times S^2]$ *class and one nodal leaf with two embedded spherical components, each in the class* E *and* $F - E$ *respectively. Also, we require that the complement of the singular leaf is a smooth foliation over* Y *which is a compact curve of genus* g *except on a single point.*

Remark 5.3. *[***Zhang16***]'s Lemma provides that for each* J *there is a singular foliation with J-hol leaves.*

Let $\mathcal{F}_{std}$ be the standard blow up foliation by J_{std}-holomorphic leaves. Note that if we blowdown the complex structure, we obtain the split complex structure on $\Sigma_g \times S^2$, and the induced foliation is the split foliation by the spheres.

Lemma 5.4. *Let* Fol_0 *be the connected component of* Fol *that contains* $\mathcal{F}_{std}$. $\mathcal{A}_\infty$ *is weakly homotopic to* Fol_0.

Proof. Observe that there is a map $\mathcal{A}_\infty \to \mathrm{Fol}_0$ given by taking J to the singular foliation of $M_g\#\overline{\mathbb{C}P^2}$ by J-spheres in class F or $F-E$. Standard arguments in [18] Ch 2.5 show that this map is a fibration with contractible fibers. Hence it is a homotopy equivalence. $\square$

Lemma 5.5. *There is a transitive action of* $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ *on* Fol_0.

Proof. Since $S^2 \setminus pt$ is compact and simply connected, each generic leaf of this foliation has trivial holonomy and hence has a neighborhood that is diffeomorphic to the product $D^2 \times S^2$ equipped with the trivial foliation with leaves $pt \times S^2$.

Since our foliation has smoothly embedded leaves and only one nodal leaf, we can find a 2-form transverse to each leaf. And the Poincaré dual of such 2-form is a smooth section, not passing through the singular point p.

Now let's take an arbitrary singular foliation $\mathcal{F}' \in \mathrm{Fol}_0$ and denote the smooth section by Σ'. We'll prove that $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ takes this foliation $(\mathcal{F}', \Sigma')$ to $\mathcal{F}_{std}, \Sigma_{std}$ where Σ_{std} is the smooth section (which is indeed J_{std}-holomorphic).

Since $\mathcal{F}'$ and $\mathcal{F}_{std}$ are in the same path connected component, there is a $\phi \in \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ sending Σ' to Σ_{std}, such that the singular leaf of $\mathcal{F}'$ goes to the singular leaf of $\mathcal{F}_{std}$ while the two singular points are identified. Now let's fix a finite covering $\{D_i, 1 \le i \le n\}$ of Σ', such that the local foliations over D_i's cover the manifold $\Sigma_g \times S^2\#\overline{\mathbb{C}P^2}$.

Then we use partition of unity for the covering $\{D_i, 1 \le i \le n\}$ of Σ', and for each local foliation, we apply a ϕ_i such that the foliation $\mathcal{F}'$ under $\phi \circ \phi_1 \circ \cdots \circ \phi_n$ agrees with the foliation $\mathcal{F}_{std}$.

Now we have the transitive action of $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ on Fol_0. Notice that this action of $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ does not necessarily preserve the leaf. $\square$

Hence there is a fibration sequence

$$\mathcal{D} \cap \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) \to \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) \to \mathrm{Fol}_0, \tag{5}$$

where $\mathcal{D}$ is the diffeomorphism preserving the leaves in the foliation $\mathcal{F}_{std}$. We denote this fiber group by $\mathcal{D}^1_g$.

Definition 5.6. *$\mathcal{D}^1_g$ is the elements in the identity component of the diffeomorphisms which fit into the commutative diagram*

$$\begin{array}{ccc} M_g\#\overline{\mathbb{C}P^2} & \stackrel{\phi}{\to} & M_g\#\overline{\mathbb{C}P^2} \\ \downarrow & & \downarrow \\ (M_g, p, F_p) & \stackrel{\phi'}{\to} & (M_g, p, F_p) \\ \downarrow & & \downarrow \\ (\Sigma_g, pt) & \stackrel{\phi''}{\to} & (\Sigma_g, pt). \end{array}$$

Here p is the intersection point $E \cap (F - E)$ of the singular fiber. And the first level of the downward arrow means that we contract the E component. We abuse notation here to still denote p for the point in M_g after contracting the curve E.

On the second level, ϕ' is a diffeomorphism of M_g keeping the point p fixed and fixing the fiber F_p passing through p fixed as a set, and preserves other leaves in the standard foliation.

The base Σ_g is the holomorphic curve B_{std} w.r.t the standard complex structure, and the map downward is obtained by firstly blow down the exceptional sphere and then projects down to the base curve.

Proposition 5.7. (1) *$\mathcal{D}^1_g$ is weakly homotopic to $G^1_{\infty,g}$.*
(2) *The group $\mathcal{D}^1_g$ is disconnected when $g \geq 2$.*
(3) *When $\mu \to \infty$, s.t. $\pi_i(G^1_{u,g}) = \pi_i(G^1_{\infty,g})$ for $i \leq min\{Cod(u)\} - 1$, and hence the groups $G^1_{u,g}$ are disconnected for $g \geq 2$.*

Proof. For statement (1), note the equation (5) fits into the commutative diagram:

$$\begin{array}{ccc} \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) & \to & \mathcal{A}_\infty \\ \downarrow & & \downarrow \\ \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) & \to & \mathrm{Fol}_0, \end{array}$$

where the map $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) \to \mathcal{A}_\infty$ is given as above by the action $\phi \mapsto \phi_*(J_{std})$. Hence there is an induced homotopy equivalence from the homotopy fiber $G^1_{\infty,g}$ of the top row to the fiber $\mathcal{D}^1_g$ of the second.

Now we prove statement (2), first note that we have the following Birman exact sequence (cf. [3]) when $(g, n), g > 0, n > 0$ is not $(1, 1)$:

$$1 \longrightarrow \pi_1(\Sigma_{g,n-1}) \longrightarrow \Gamma(g, n) \longrightarrow \Gamma(g, n-1) \longrightarrow 1.$$

Here $\Gamma(g, n)$ means the mapping class group of Σ_g fixing n points.

In our case, we are looking at the case $n = 1$

$$1 \longrightarrow \pi_1(\Sigma_g) \longrightarrow \Gamma(g, pt) \longrightarrow \Gamma(g) \longrightarrow 1.$$

We first up each point along a loop ($t \in [0, 2\pi]$) which is homologically non-trivial on the Σ_g. This give rise to a loop of complex structures J_t, where $J_0 = J_{std}$. Note that we can use a path ϕ_t in $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ to push J_0, such that $\phi_t \circ J_0 = J_t$. Note that this J_t family gives rise to a loop of foliations, which pushing the marked point p along a homological non-trivial circle on the standard base Σ_g for time $t \in [0, 2\pi]$, and denote this loop of foliation $f_t, t \in [0, 2\pi]$. Note that ϕ_t in $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ pushes the standard foliation along this loop. We now show that this loop gives rise to an element that is not isotopic to id in $\mathcal{D}^1_g$. Suppose not, by path lifting of the fibration 5, we would have a leaf-preserving element in $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$, so that it is isotopic to identity through a path in $\mathcal{D}^1_g$. Furthermore, this path pushes the given foliation along the lifting of the loop $f_t, t \in [0, 2\pi]$. Now apply diagram in definition 5.6, we would have an isotopy, this would give an isotopy of (Σ_g, p), connecting the time 2π diffeomorphism to identity. This is an contradiction against the Birman exact sequence. Hence statement (2) holds.

Statement (3) follows from the stability Theorem 1.2.

□

Remark 5.8. *When $g = 0$, one can blow up $S^2 \times S^2$ at k points with equal sizes. It is shown in [10] that when $k \leq 3$, $G^k_{u,0}$ is connected for all ω. When $k > 3$, $\pi_0 G^k_{u,0}$ is a braid group of k strands on spheres (cf. [9]). This follows the same pattern as $\mathrm{Diff}(S^2, k)$, which is the diffeomorphism group of S^2 fixing k points. In particular, when we take the one point blowup of $S^2 \times S^2$, all arguments in the current paper apply here. Even though we have a model for the colimit group, we are not able to show it's connected, but the loop that appears in our proof of Proposition 5.7 (2) must give rise a trivial mapping class since $\mathrm{Diff}(S^2, 1)$ is connected.*

Remark 5.9. *It remains an open question whether the colimit group for $g = 1$ is connected or not. Shevchishin-Smirnov showed in [**SS17elliptic**] that there exists a symplectomorphism called elliptic twist along a (-1) torus on the minimal ruled surface or its one point blowup. They further showed that when $\mu \to \infty$ the elliptic twist is isotopic to identity, and a non-trivial mapping class appears in the first chamber in our Figure 2. The two potential non-trivial symplectic mapping classes, elliptic twists as in [**SS17elliptic**] and the loop in Proposition 5.7 (2), fail to appear in the colimit group. In [**SS17elliptic**] this is because the (-1) tori has a positive area. In our case, this is because adding one puncture to a torus does not change* *its mapping class group. Hence we still don't know the connectedness of the group D^1_1, and we conjecture that it is connected.*

Remark 5.10. Implicit in the above argument is the following description of the map $G^1_{\infty,g} \to \mathcal{D}^1_g$. Let $\mathcal{J}_\mu$ denote the space of all almost complex structures compatible with ω_μ. Since the image of the group G_u under the map $\mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) \to \mathcal{A}_u$ is contained in $\mathcal{J}_\mu$ there is a commutative diagram

$$\begin{array}{ccc} & & \mathcal{D}^1_g \\ & & \downarrow \\ G_u & \xrightarrow{\iota} & \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2}) \\ \downarrow & & \downarrow \\ \mathcal{J}_\mu & \longrightarrow & \mathrm{Fol.} \end{array}$$

Because $\mathcal{J}_\mu$ is contractible, the inclusion $\iota : G_u \to \mathrm{Diff}_0(M_g\#\overline{\mathbb{C}P^2})$ lifts to a map $\tilde{\iota} : G_u \to \mathcal{D}^1_g$. Now take the limit to get $G_\infty \to \mathcal{D}^1_g$.

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