PENNER'S PSEUDO-ANOSOV MAPS VIA HALF TWISTS

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Abstract. We'll review various constructions of pseudo-Anosov maps, in particular, Penner's construction using (full) Dehn twists. Also, inspired by Verberne's recent work of pseudo-Anosov maps that is not of Penner's types, we provide a counterexample to a version of Penner's conjecture for half twists.

Contents

1. Introduction

A surface is a two-dimensional manifold. That is, a topological space that locally homeomorphic to $\mathbb{R}^2$.

The following are common examples.

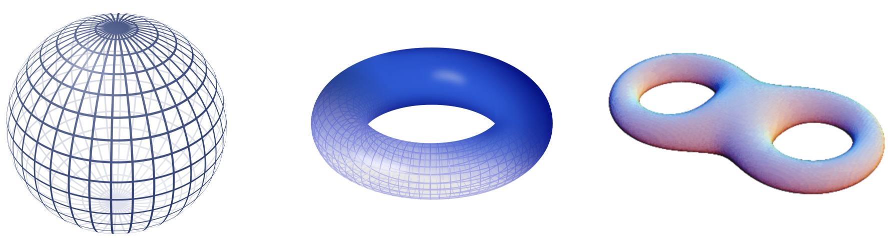


Figure 1. A sphere [3], a torus [5], and a genus 2 surface [9].

Let $S_{g,n}$ be a surface of genus g with n punctures. The mapping class group, $(S_{g,n})$, is the group of isotopy classes of orientation-preserving homeomorphisms of $S_{g,n}$. The Nielsen-Thurston classification of elements states that each element in $(S_{g,n})$ is either periodic, reducible, or pseudo-Anosov. Homeomorphisms are of utmost interest to topologists, and pseudo-Anosov maps are the more complicated and important ones. Now we introduce the definition.

Definition (Homeo$^+(S_{g,n})$, Mapping Class, Mapping Class Group, cf. [6]). *Given a surface $S_{g,n}$,*

- Homeo$^+(S_{g,n})$ *is the set of orientation-preserving homeomorphisms from S to itself.*
- *A* mapping class *of S is a homotopy class of homeomorphisms from $S_{g,n}$ to itself.*
- *The* mapping class group *of $S_{g,n}$ is the group of mapping classes of $S_{g,n}$.*

Theorem 1.1 (Nielson-Thurston Classification of Mapping Classes cf. [6]). *Given a surface $S_{g,n}$, let h be a homeomorphism from $S_{g,n}$ to itself. The at least one of the following is true:*

- *h is periodic, i.e. some power of h is the identity.*
- *h is reducible, i.e. h preserves some finite union of disjoint simple closed curves on S.*
- *h is pseudo-Anosov, i.e. no power of h fixes any curve on S.*

Note that the above theorem is a re-phasing of Thurston's proof of the Nielsen-Thurston classification. It provided us with the definition of pseudo-Anosov mapping classes. Thurston defined an element $f \in (S_{g,n})$ to be pseudo-Anosov if there is a representative homeomorphism ϕ, a number $\lambda > 1$ and a pair of transverse measured foliations $\mathcal{F}^\sqcap$ and $\mathcal{F}^\int$ such that $\phi(\mathcal{F}^\sqcap) = \lambda\mathcal{F}^\sqcap$ and $\phi(\mathcal{F}^\int) = \lambda^{-1}\mathcal{F}^\int$. λ is called the *stretch factor* (or dilatation) of f, $\mathcal{F}^\sqcap$ and $\mathcal{F}^\int$ are the *unstable foliation* and *stable foliation*, respectively, and the map ϕ is a *pseudo-Anosov homeomorphism.*

It is known since then that pseudo-Anosov maps are generic and of great importance not only in 2-dimensional theory but also in 3-dimensional hyperbolic manifolds and the geometrization conjecture. Thurston and Penner provided some constructions and raised the following question still remains open:

Question 1.2. *Is there an explicit construction for all pseudo-Anosov mapping classes?*

To better understand the question, we introduce an important mapping class called Dehn twists that belongs to the reducible mapping classes in the Nielson-Thurston classification:

Definition ([8]). *Consider a simple closed curve described above. We say that such a curve γ separates punctures k and l from $S_{0,n}$ if one of the subsurfaces obtained by cutting along γ contains only the punctures k and l, and the other subsurface contains the remaining punctures. Denote the curve separating puncture j and $j-1 \mod n$ by α_j. Define the half-twist associated to puncture j, denoted D_j, as the half-twist around α_j. Two subsequent half-twists, D_j^2, is called a Dehn twist.*

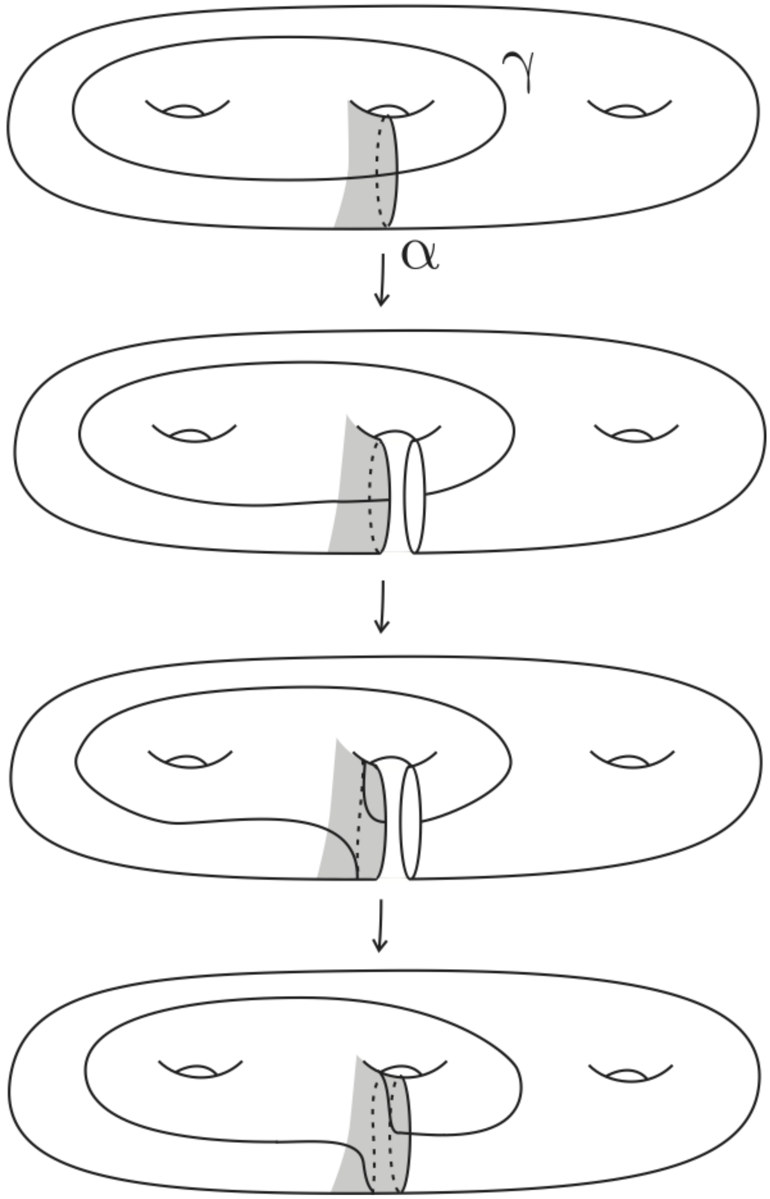


FIGURE 2. Demonstration of Dehn Twist

Theorem 1.3 (Dehn, cf. [6]). *All mapping classes of $S_{g,n}$ can be written as products of Dehn twists.*

2. Penner's construction, examples and conclusion

It is widely known in the literature that there is a standard construction of pseudo-Anosov maps called Penner's construction.

Theorem 2.1 (Penner's Construction [10])**.** *Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be multicurves on a surface S in minimal position such that $(A \cup B)^c$ is a union of disks and once punctured disks. Then any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov, provided that all $n + m$ Dehn twists appear in the product at least once.*

This construction is fairly general, so Penner made the following conjecture.

Conjecture (Penner, 1988 [10])**.** *Every pseudo-Anosov mapping class has a power that arises from Penner's construction.*

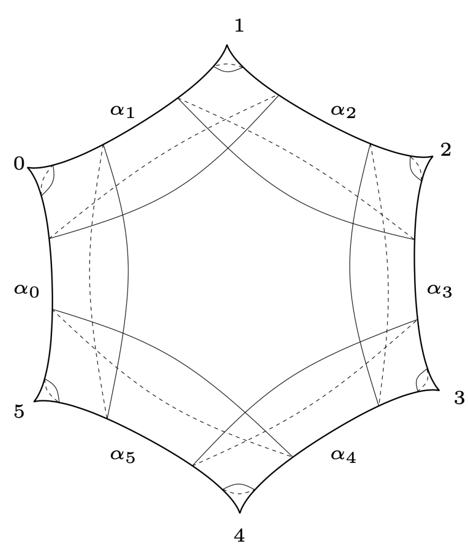
However, this conjecture has been disproved and it has been shown that there exist pseudo-Anosov maps such that no power of them comes from Penner's construction for most surfaces (Shin & Strenner, 2015) [7].

A new construction of a pseudo-Anosov Map via full twists. Verberne (2019) presents a method to construct pseudo-Anosov maps, some of which not coming from Penner's construction. Here is the first example:

Theorem 2.2 (Verberne, 2019, [8])**.** *On $S_{0,6}$, the composition of Dehn twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$*

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2$$

is a pseudo-Anosov map, where D_i is a half Dehn twist around α_i.



2.1. **A pseudo-Anosov Map Visualized in Polygonal Representation.** In fact, no curve on the surface is preserved under ϕ^n for all $n \in \mathbb{N}$. This is a characterization of pseudo-Anosov maps.

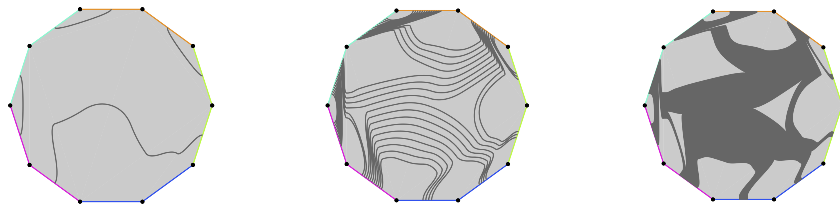


FIGURE 3. A curve, its image under ϕ, and that under ϕ^2 on an six-times-punctured sphere.[2]

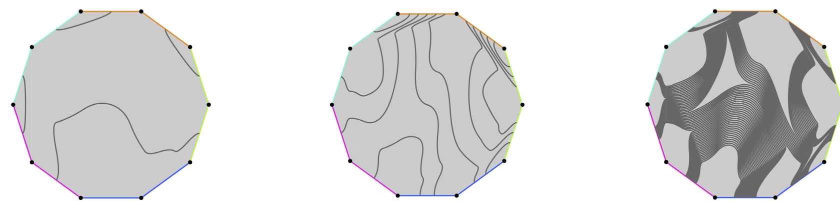


FIGURE 4. A curve, its image under ϕ^4, and that under ϕ^8 on an six-times-punctured sphere.[2]

2.2. **Pseudo-Anosov Maps Coming from Half Twists.** There is an analog of Penner's construction for half twists.

Conjecture 2.3. *Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be multicurves (not necessarily simple closed) on a surface $S_{g,n}$ in minimal position such that $(A \cup B)^c$ is a union of disks and once punctured disks. Then any product of positive half twists about a_j and negative half twists about b_k is pseudo-Anosov, provided that all $n+m$ Dehn twists appear in the product at least once.*

Example. *On $S_{0,6}$, the composition of half Dehn twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3,\ \alpha_4, \alpha_5$*

$$\phi = D_0 D_3^{-1} D_1 D_4^{-1} D_2 D_5^{-1}$$

is a pseudo-Anosov map.

There is a statement that is a direct analog of Penner's construction using half twists on punctured spheres.

However, the direct translation (Conjecture 2.3) of the full twist version does not hold. Hence we ask Question 1 and will further probe in this direction.

Counterexample. *On $S_{0,6}$, the composition of half twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3,\ \alpha_4, \alpha_5$*

$$\phi = D_0 D_3 D_1 D_4^{-1} D_2 D_5^{-1}$$

is reducible.

Note that this is to say that Conjecture 2.3 is not ture.

Discussions and Further Questions

Notice that Verberne indeed gave a more general construction for pseudo-Anosov $S_{0,n}$

Theorem 2.4 (Verberne, [8])**.** *Consider the surface $S_{0,n}$. Let $\{\mu_i\}_{i=1}^k$, for $1 < k < n$, be an evenly spaced partition of the punctures of $S_{0,n}$. Then*

$$\phi = \prod_{i=1}^{k} D_{\mu_i}^{q_i} = D_{\mu_k}^{q_k} \dots D_{\mu_2}^{q_2} D_{\mu_1}^{q_1},$$

where $q_j = \{q_{j_1}, \dots q_{j_l}\}$ are tuples of integers greater than one, is a pseudo-Anosov mapping class.

Also, the constructions of Shin & Strenner [7] works for more general surfaces $\S_{g,n}, g, n >> 0$.

These constructions do not give every pseudo-Anosov map for an arbitrary surface. In particular, there are two questions we would like to probe in the future.

Question 2.5. *In general, how to construct pseudo-Anosov maps using half twists on punctured spheres?*

Flipper can guide our intuition. For example, a combination of half-twists gives examples of pseudo-Anosov maps, but we still need to discover whether this is different from the other constructions. We are looking for a more precise analog of Penner's construction using half twists.

Question 2.6. *Does there exist a construction using train track methods ([1]) such that every pseudo-Anosov map comes from it?*

This will not easily be answered. It is already known that Penner's construction does not give all pseudo-Anosov maps, but maybe one or more of the current constructions give all of them.

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