CIRCLE ACTIONS AND ISOTOPY ON SPACE OF POLYGONS

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ABSTRACT. In this note, we study the moduli space of polygons in $\mathbb{R}^3$ and their relatives. The observation in [19] says that if there is a circle action in the complement of a Lagrangian sphere, then the square Dehn twist is isotopic to identity. We use a projective twists version of this observation to provide several families of such examples in polygon spaces, and related Gelfand-Celtin systems.

CONTENTS

1. INTRODUCTION

The symplectomorphism group $Symp(M, \omega)$ of a symplectic manifold (M, ω) consists of all symplectomorphisms, which are diffeomorphisms between the manifold and itself that preserve the symplectic structure.

The question of whether a given symplectomorphism is isotopic to the identity symplectomorphism is indeed a specific instance of the symplectic isotopy problem. In this case, the problem is to determine whether there exists a smooth path of symplectomorphisms connecting the given symplectomorphism to the identity symplectomorphism. Motivated by this problem, exotic symplectomorphism generalizing Dehn twists are discovered.

In diemnsion 4, significant progress has been made However, in higher dimension, it is still mysterious and the symplectic isotopy problem stands as a fundamental challenge. To tackle this challenge and exclude exotic symplectomorphisms, noteworthy progress was made in [13] with the following result:

Theorem 1.1. *For any symplectic toric manifold (M^4, ω), if a symplectomorphism $f \in Symp(M, \omega)$ acts trivially on the second homology group $H_2(M^4, \mathbb{Z})$, then it is symplectically isotopic to the identity.*

This theorem paves the way for an intriguing conjecture that looms large in the field:

Conjecture 1.2. *In dimensions $2k > 4$, we conjecture that any toric manifold M^{2k} possesses a connected subgroup $Symp_h(M, \omega)$ of the symplectomorphism group, where $Symp_h(M, \omega)$ refers to the subgroup that acts trivially on homology.*

The complexity of this conjecture is evident, as it remains a formidable challenge to prove in higher dimensions. Even in the case of $2k = 6$, our knowledge is limited. To make progress, we embark on a journey of exploration, starting with accessible examples in higher dimensions, specifically focusing on the moduli spaces of spatial polygons as symplectic manifolds.

1.1. **Moduli space of spatial polygons.** A polygon in $\mathbb{R}^3$ is determined by its vertices $v_1, \ldots, v_n$ and its oriented edges $e_1, \ldots, e_n$. For any vector $r = (r_1, \ldots, r_n) \in \mathbb{R}^n_+$, M_r will denote the space of polygons with edge lengths $r_1, \ldots, r_n$ modulo rotations and translations. We consider the moduli space of those objects:

Definition 1.3. *Let $r = (r_1, \cdots, r_n)$ be an n-tuple of positive numbers. Consider spheres $S_{r_i} \to \mathbb{R}^3$, each of radius r_i. Define the addition map $\mu : S_{r_1} \times \cdots \times S_{r_n} \to \mathbb{R}^3$ as $\mu(e_1, \cdots, e_n) = e_1 + \cdots + e_n$. The pre-image $\mu^{-1}(0)$ is invariant under the diagonal action of $SO(3)$. The quotient $M(r, n) = \mu^{-1}(0)/SO(3)$ forms a manifold of dimension $2n - 6$.*

These manifolds, denoted as $M(r, n)$ or $M(r)$, are known as the moduli spaces of spatial polygons, or polygons in three-space $\mathbb{R}^3$. Each point of $M(r)$ is a polygons, represented as edge vectors $(e_1, \cdots, e_n) \in \Pi S_{r_i}$, situated in three-space, allowing for orientation-preserving rigid motions of $\mathbb{R}^3$. It's worth noting that these polygons may intersect themselves. In [11], reveals the existence of natural symplectic structures, bending flows, torus actions, and circle actions on $M(r)$, induced from $S_{r_1} \times \cdots \times S_{r_n}$. Different weight $r = (r_1, \cdots, r_n)$may give different topology of $M(r)$. Also, notice that the rescaling of the vector $r = (r_1, \cdots, r_n)$ preserves the topology and gives rise to a rescaling of the symplectic structure.

Further, in [8], the topology of these polygons, including their rational homology ring, is comprehensively explored. The space of $M(r, n)$ is a polyhedron and there is a chamber structure where the topology only change when crossing the wall. In terms of birational geometry, those spaces $M(r, n)$ are $\mathbb{C}P^{n-3}$ blowup at points or linear subspaces. The generic $M(r, n)$ is also related to the canonically compactified Deligne-Mumford space of stable n-pointed projective lines $\overline{M_{0,n}}$, where the explicit birational map is given by [9]. We will detail their topology in section 2.

1.2. **Isotopy Results on Projective twists.** We unveil several results that shed light on the intricacies of the symplectic isotopy problem in dimension 6.

As a first step, we consider the moduli space of plane polygons, and they are natural Lagrangian submanifolds in $M(r)$. The topology of those submanifolds can vary, but in certain special cases, they are projective spaces ($\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{H}P^n$ for example), where we can define Dehn twist-like mappings, called projective twists (see Definition 4.2 for more details) along those Lagrangian submanifolds. When there are natural circle action on the complement of the Lagrangian, we have

Proposition 1.4. *Let $L \subset M$ be a projective Lagrangian, if there is a circle action on $M \setminus L$, then the square projective twists along Lagrangian submanifold L is symplectically isotopic to identity in M.*

Let $\vec(r) \in \mathbb{R}^+_m$ be a generic vector $(1, \cdots 1, x)$, $x \in (\lfloor \frac{m-1}{2} \rfloor - 1, m-1)$, and $M(\vec{r}, m)$ be the corresponding polygon spaces. We are able to prove that

Theorem 1.5. *The square of certain projective twists along Lagrangian $\mathbb{R}P^n$ and $\mathbb{C}P^n$ in the above family of $M(r, m)$ are isotopic to identity.*

We give two families of polygon space, and Lagrangian $\mathbb{R}P^n$ or $\mathbb{C}P^n$ in those manifolds as explicit examples. However, a complete list is not availabel at this moment. Meanwhile, we find there are examples related to Gelfand-Cetlin systems where Propsition 1.4 applies, see section 4.2 for more details.

The above corollary can be regarded as a generalization of Theorems on Seidel's remark of Dehn twist along Lagrangian spheres of small rational manifolds to higher dimensions. Notice that Seidel's remark states that Dehn twists along certain $S^2 = \mathbb{C}P^1$'s are isotopic to identity in $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \leq 4$. However, there is another difficulty (Donaldson's generation question) toward Conjecture 4.1, which is whether the full symplectic Torelli group $\pi_0(Symp_h(M, \omega))$ is generated by Lagrangian Dehn twists. At this point, only in dimension 4, the question has an affirmative answer for positive rational surfaces by [13] and in higher dimensions, very little is known.

These examples demonstrate the diversity and complexity of the symplectic world, offering tantalizing challenges for future research.

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2. Topology and torus actions of polygon spaces

$M(r)$ is non-empty if and only if all permutations of the inequality $r_1 \leqslant r_2 + \cdots + r_n$ hold. Thus, the space of valid r is a polyhedron in $\mathbb{R}^n$. Since there are n-inequalities, the polyhedron is actually a cone on a simplex. For example, if we choose to normalize so Σr_i is a constant, then the region of r_i is a simplex. We will often refer to e_i for the corresponding edges.

This simplex is cut by a number of hyperplanes corresponding to the r not satisfying our standing assumption. These hyperplanes cut the region of valid r into sub-regions called chambers. Moreover, the bending flow in [11] shows that there is Hamiltonian torus action on the polygon space for a generic weight vector r, which we will detail in the section

The topology of $M(r)$ is constant in each chamber. One nice way to see this is that the corresponding moduli of points are isomorphic. One can just change the weights from one r to another r_0 in the same chamber, and this will never result in a point with total weight $\frac{1}{2}$ or more. Indeed, one can linearly or smoothly interpolate between r and r_0. If the weights at some point reached $\frac{1}{2}$ at a point, this would prove that this intermediate value of r_0 did not lie in the same chamber.

Depending on the weight we have a computation of the rational homology ring in [8]. Here we detail how the chambers are determined, using the inequalities of short subsets.

Let $r = (r_1, \ldots, r_m) \in \mathbb{R}^m_+$. A subset $J \subset \{1, \cdots, m\}$ is considered "short" if the sum of its components is less than or equal to the sum of the components not in J. In other words, J is short if and only if the sum $\sum_{j=1}^m (-1)^{\chi_J(i)} r_j$ is greater than or equal to zero, where χ_S is the characteristic function of set S. For instance, the empty set is short, and singletons are short if and only if $M(r) \neq \emptyset$. More generally, a set S is short if and only if there exist configurations in $M(r)$ where all edges in S are parallel. It's important to note that these equalities cannot hold if r is assumed to be generic. We define the set $\mathcal{S} := \mathcal{S}(r) := \{J \subset \{1, \ldots, m\} \bigm| J \text{ is short}\}$.

Let $r \in \mathbb{R}^m_+$, and let $\mathcal{S} := \mathcal{S}(r)$. For $k \in \{1, 2, \ldots, m\}$, we introduce the subposet $\mathcal{S}_k$ of $\mathcal{S}$ as $\mathcal{S}_k = \mathcal{S}_k(r) := \{J \subset \{1, \ldots, m\} - \{k\} \bigm| J \cup \{k\} \in \mathcal{S}\}$. In the subsequent sections, we provide the Poincaré polynomial and presentations of the cohomology ring of $M(r)$ in terms of $\mathcal{S}_m$. Further, the diffeomorphism type of $M(r)$ is determined by any of the subposets $\mathcal{S}_k$.

Let P_X be the Poincaré polynomial of a manifold X.

Proposition 2.1. *The various polygon spaces are even-cohomology spaces, and their Poincaré polynomials are given by*

$$P_{M(r)} = \frac{1}{1-t^2} \sum_{J \in \mathcal{S}_m} (t^{2|J|} - t^{2(m-|J|-2)})$$

Those spaces $M(r)$ are all irrational to $\mathbb{C}P^{m-3}$, by [10]. Moreover, their topology up to diffeomorphism is completely determined by their integral homology, see [4, Theorem 3].

2.1. **Bending flow and torus actions.** Kapovich and Millson [11] proved that any triangulation of the standard n-gon yields a Hamiltonian action of T^{n-3} on $M(\vec{r}, r)$ where the angle θ_i acts by folding the polygon around the i^{th} diagonal of the triangulation (called a bending flow in symplectic geometry and a polygonal fold or crankshaft move in random polygons). The induced moment map $\mu : M(\vec{r}, n) \to \mathbb{R}^{n-3}$ records the lengths l_i of the diagonals in the triangulation.

Here we give more detail, and recall that a polygon is degenerate when it lies on a line. The moduli space $M(r)$ forms a smooth manifold if and only if the lengths vector r satisfies a genericity condition, meaning for each $I \subset 1, \dots, n$, the quantity $\Delta_I(r) := \sum_{i \in I} r_i - \sum_{i \in I^c} r_i$ is nonzero.

Take a generic $r \in \mathbb{R}^m_+$. For any polygon P in $M(r)$ having edges $\vec{e}_1, \dots, \vec{e}_m$ and vertices $v_1, \dots, v_m$, we can define a system of $m-3$ non-intersecting diagonals $\vec{d}_1, \dots, \vec{d}_{n-3}$ starting from the first vertex to the other non-adjacent vertices, so $\vec{d}_i(P) = \vec{e}_1 + \cdots + \vec{e}_{i+1}$.

The lengths of these $n-3$ diagonals $(l_1, \dots, l_{n-3}) : M(r) \to \mathbb{R}^{n-3}$ mapping $P \mapsto (|\vec{d}_1(P)|, \dots, |\vec{d}_{n-3}(P)|)$ give continuous functions on $M(r)$ that are smooth where nonzero. Their image forms a convex polytope $\Delta \subset \mathbb{R}^{n-3}$, containing points $(l_1, \dots, l_{n-3})$ satisfying triangle inequalities:

$$\begin{aligned} r_{i+2} &\leqslant l_i + l_{i+1} \\ l_i &\leqslant r_{i+2} + l_{i+1} \\ l_{i+1} &\leqslant r_{i+2} + l_i \end{aligned}$$

for $i = 0, \dots, n-3$, where $l_0 = r_1$ and $l_{n-2} = r_n$.

These l_i functions generate Hamiltonian flows called bending flows. For a given diagonal $\vec{d}_i$, the associated circle action on the dense open set $l_i \neq 0 \subset M(r)$ bends the first $i+1$ edges along $\vec{d}_i$ at constant speed while the rest are fixed. Combining the actions from the $(n-3)$ diagonals yields a toric T^{n-3} action on $\{l_i \neq 0, i = 1, \cdots, n-3\} \subset M(r)$, and the symplectic form is given by $\Sigma_i d\theta_i \wedge d(l_i)$.

Moreover, this chamber structure and torus action is related to the GIT quotient $Gr(2,n)//U^n(1)$, where referred to the symplectic version of Gelfand-MacPherson correspondence. When the weight vector crosses a wall given by a degenerated polygon, the topology and symplectic structure changes follows the way of variation of GIT. This is described in [15] using the work of Guillemin–Sternberg [7].

Here we recall Theorem 4.1 of [15]

Theorem 2.2. *Given two weight vectors in different chambers, $\vec{r}_0 \in \mathcal{C}_0$ and $\vec{r}_1 \in \mathcal{C}_1$ separated by the wall $\Delta I^p(r) := \sum_{I_p} r_i - \sum_{I_q} r_i = 0$, where $I_q = I \setminus I_p$. Let $M_{I_p}(\vec{r})$ be themoduli space of polygons obtained by letting all e_i's proportional to each other for $i \in I_p$. The topology of the moduli space of polygons $M(\vec{r})$ are related by blowing up $M_{I_p^c}(r_0) \simeq \mathbb{CP}^{p-2}$ and blowing down the projectivization of normal bundle of $M_{I^p}(r_1) \simeq \mathbb{CP}^{q-2}$.*

Moreover, we remark that

Remark 2.3. *The blowups and blowdowns from $\mathbb{C}P^n$ can be done S^1-equivariantly (cf. [7]), where the circle action is the diagonal action of the torus acting on $\mathbb{C}P^n$. Notice that in there is a birational geometry description for (projective) toric varieties using GIT cf. [17].*

2.2. **Explicit examples.** We focus some families with long r_n, compute the cohomology of $M(r)$, and hence determine their diffeomorphism types.

First, let's consider the simplest cases:

Example 2.4 (=Example 10.1 in [8])**.** *Suppose $\mathcal{S}_m(r) = \{\emptyset\}$, for instance, when $r = (1, \ldots, 1, m-2-\epsilon)$. Indeed a generic $r = (1, \ldots, 1, k)$ with $k \in (m-3, m-1]$ works the same. As $\mathcal{S}_m(r) = \{\emptyset\}$, the expression of the Poincaré polynomial $P_{M(r)}$ given in Theorem 2.1 reduces to a single term:*

$$P_{M(r)} = \frac{1-t^{2(m-2)}}{1-t^2} = 1 + t^2 + \cdots + t^{2(m-3)}$$

which matches the Poincaré polynomial of $\mathbb{C}P^{m-3}$. It follows that $M(r)$ is diffeomorphic to the complex projective space $\mathbb{C}P^{m-3}$ in this case.

Another simple example is

Example 2.5 (=Example 10.2 of [8])**.** *Let's consider a generic $r = (\epsilon, \ldots, \epsilon, 1, 1, 1)$ with $(m-3)\epsilon < 1$ (or after rescaling, for example, $r = (1, \ldots, 1, m-2, m-2, m-2)$). Then the Poincaré polynomial is given by*

$$P_{M(r)} = \prod_{i=1}^{m-3} (1 + t^2).$$

Therefore, $M(r)$ in this case is symplectomorphic to $\prod_{i=1}^{m-3} S^2_{r_i}$.

Next, we have the blowup of $\mathbb{C}P^N$ at a certain number of points. Notice that If X is of dimension n and B_pX the blow-up at a point $p \in X$ then $P_{B_pX}(t) = P_X(t) + t^2 + \cdots + t^{2n-2}$. More generally, blowing-up along a linear subspace P^k add $t^{2k+2} + \cdots + t^{2n-2-2k}$ to the Poincaré polynomial.

Example 2.6. *We note that the initial case mentioned, $r_m = (1, 1, \cdots, 1, m-3)$. There is one short set $\emptyset$, $m-1$ of short sets being $\{1\}$, and no other short sets. The Poincaré polynomial is given by*

$$P_{M(r)} = \frac{1-t^{2(m-2)}}{1-t^2} = 1 + t^2 + \cdots + t^{2(m-3)} + (m-1)(1 + t^2 + \cdots + t^{2(m-4)}).$$

This leads to $M(r)$ being a smooth manifold diffeomorphic to $\mathbb{C}P^{m-3}\sharp(m-1)\overline{\mathbb{C}P^{m-3}}$, a blowup of $\mathbb{C}P^{m-3}$ at $m-1$ points in general position.

A special case worth mentioning is the **regular pentagon***: $r = (1, 1, 1, 1, 1)$. In this scenario, $M(r)$ is a smooth manifold diffeomorphic to $(S^2 \times S^2)\sharp 3\overline{\mathbb{C}P}^2 \simeq \mathbb{C}P^2\sharp 4\overline{\mathbb{C}P}^2$.*

Notice that there are other chambers where the topology of $M(r)$ are blowups of $\mathbb{C}P^N$ at linear subspaces, instead of points. Moreover, the Euler number of the blowup could be larger than the one given in Example 2.6. However, we highlight this example because of the following connection to the Gelfand-MacPherson correspondence.

Remark 2.7. *We have the following equivalence of GIT quotients called Gelfand-MacPherson correspondence:*

$$(P^{d-1})^n /\!\!/ PGL_d(\mathbb{C}) \cong Gr(d,n) /\!\!/ T_{U(n)} \cong Gr(n-d,n) /\!\!/ T_{U(n)} \cong (P^{n-d-1})^n /\!\!/ PGL_{n-d}(\mathbb{C}).$$

When $n = d+2$, we have the following $Conf(\mathbb{C}P^n, n+3) \simeq Conf(\mathbb{C}P^1, n+3)$.

Notice that when $k < n+3$, $Conf(\mathbb{C}P^n, n+3)$ is trivial, because $PGL_{n+1}(\mathbb{C})$ acts $(n+2)$ transitively on $\mathbb{C}P^n$. Hence $(n+2)$ blowup at $\mathbb{C}P^n$ is a critical value, which is analogous to Seidel's observation in Example 2.13 of [19].

One observation (cf. Section 3 of [11]) is that there is a bending flow that makes $M(r)$ almost toric. A blowup of $\mathbb{C}P^n$ at a small number of points in general position often admits a torus action. Note that adjusting the weight, one can obtain blowup at $\mathbb{C}P^n$ at less than $(n+2)$ points. This means that polygon spaces provide a large class of examples for Conjecture 1.2.

Example 2.8. *Now let m be $2n+1$, and a generic $r_m = (1, 1, \cdots, 1, N+\epsilon)$, where $N \in [n, m-2)$. There is one short set $\emptyset$, $m-1$ of short sets being $\{1\}$, and C^j_{m-1} of* *short sets J with $j = |J| \leq n$. No other short sets exists. The Poincaré polynomial is given by*

$$P_{M(r)} = \sum_{i=0}^{m-N} \binom{m-1}{i} \frac{1 - t^{2(m-2i-2)}}{1 - t^2}.$$

Note that this is a blowup of $\mathbb{C}P^(n-3)$ at points and linear subspaces.

Note that for m even, this also works. For a general m, a generic vector $r_m = (1, 1, \cdots, 1, \lfloor \frac{m-1}{2} \rfloor - 1 + \epsilon)$ gives the above manifolds and the counterpart for m even.

We note that unlike the real dimension 4 case, where one have both $\mathbb{C}P^2$ and $\mathbb{C}P^1 \times \mathbb{C}P^1$ as the minimal model, this is no longer true in real dimension 6 or higher. Let $B_r(P^1)^n$ be the blow-up of $(P^1)^n$ at r points, and $B_s P^n$ the blow-up of P^n at s points, then equality of Poincaré polynomial implies that

$$\begin{aligned}(1+t^2)^n + r(t^2 + \cdots + t^{2n-2}) &= 1 + t^2 + \cdots + t^{2n} + s(t^2 + \cdots + t^{2n-2}) \\ \implies r + n - 1 &= s + 1 \\ \text{and} \quad r + \binom{n}{2} &= s + 1\end{aligned}$$

and we easily see that this system that has no solutions if $n \geqslant 3$.

However, we do have the following family that realizes as blowup of $\mathbb{C}P^k \times \mathbb{C}P^k$ at points and linear subspaces:

Example 2.9. *Notice that the Poincaré polynomial of $\mathbb{C}P^k \times \mathbb{C}P^k$ is $(1+t^2+\cdots+t^{2k})^2$, and multiply this by $1-t^2$ we get $1 + t^2 + \cdots + t^{2k} - t^{2k+2} - \cdots - t^{4k-2} = (1 - t^{4k+2}) + t^2 - t^{4k-4}) + \cdots + (t^{2k+2} - t^{2k})$. Hence $m = 2k+3$, or $k = n-1$, we have $P_{M(r)} = P_{\mathbb{C}P^n \times \mathbb{C}P^n} + \sum_{i=1}^{m-N} [\binom{m-1}{i} - 1] \frac{1-t^{2(m-2i-2)}}{1-t^2}$.*

Note that for all $N \in [n, m-3]$, $[\binom{m-1}{i} - 1]$ is positive. Then $M(r)$ is a smooth manifold diffeomorphic to $\mathbb{C}P^k \times \mathbb{C}P^k$ blowup at linear subspaces of various dimension.

3. Dehn twists and circle actions in dimension 4

This section follows [19], and we recall the construction of Dehn twist along Lagrangian spheres and the isotopy results in dimension 4.

Note in the case of dimension 2, the moduli spaces $M(r, 4)$ are all homeomorphic to $S^2 = \mathbb{C}P^1$.

Moving to dimension 4, $M(r, 5)$ exhibits diverse topological possibilities, including $\mathbb{C}P^2$, $S^2 \times S^2$, and $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, where $n \leqslant 4$. The symplectic structures on these manifolds are heavily influenced by their sizes, a topic we will delve into further in our discussion. It's noteworthy that all of these manifolds admit Hamiltonian torus actions. However, this is not the case when we venture into dimensions 6 and beyond.

In this section, we will explore various isotopy problems in these spaces, many of which have been addressed in [19] in dimension 4. These problems serve as valuable models for understanding results in higher dimensions. We will also draw comparisons between these results and the corresponding cases involving domains later.

Let's begin by briefly revisiting the definition of Dehn twists along Lagrangian spheres.

The construction of four-dimensional Dehn twists is a standard procedure, as described in [1, 20]. We will delve into the details, as they serve as the foundation for our subsequent discussions. Consider the symplectic manifold T^*S^2 equipped with its standard symplectic form ω. In coordinates, we have:

$$T^*S^2 = \{(p,q) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle p, q\rangle = 0,\, ||q|| = 1\}, \quad \omega = dp \wedge dq.$$

This manifold comes equipped with an $O(3)$-action induced from the action on S^2. Perhaps less intuitively, the function $h(p,q) = ||p||$ induces a Hamiltonian circle action σ on the complement of S^2 in T^*S^2, defined as follows:

$$\sigma_t(p,q) = \left(\cos(t)p - \sin(t)||p||q, \cos(t)q + \sin(t)\frac{p}{||p||}\right).$$

For π-rotation, σ_π corresponds to the antipodal map $A(p,q) = (-p,-q)$. However, for $t \in (0;\pi)$, σ_t does not extend continuously over the zero-section. Geometrically, with respect to the round metric on S^2, σ can be thought of as the "normalized geodesic flow," which transports each tangent vector at unit speed along the geodesic originating from it. This existence is based on the remarkable property that all geodesics on S^2 are closed, with precisely the same period.

This construction extends seamlessly to Lagrangian rank one symmetric spaces, as their normalized geodesic flow is periodic with a period of 2π.

Now, let's introduce a function $r : \mathbb{R} \to \mathbb{R}$ satisfying $r(t) = 0$ for $t \gg 0$ and $r(-t) = r(t) - t$. The Hamiltonian flow generated by $H = r(h)$ can be expressed as $\phi_t(p,q) = \sigma_{t\, r'(||p||)}(p,q)$. Given that $r'(0) = 1/2$, the time-2π map can be smoothly extended over the zero-section as the antipodal map. This gives rise to a compactly supported symplectic automorphism of T^*S^2, denoted as:

$$\tau(p,q) = \begin{cases} \sigma_{2\pi\, r'(||p||)}(p,q) & p \neq 0, \\ (0,-q) & p = 0 \end{cases}$$

This symplectic automorphism is commonly referred to as a "model Dehn twist." To apply this local model in a specific geometric context, consider a Lagrangian sphere $L \subset M$ within a closed symplectic four-manifold. Choose an identification $i_0 : S^2 \to L$. By virtue of the Weinstein neighborhood theorem, i_0 extends to a symplectic embedding:

$$i : T^*_{\leqslant\lambda} S^2 \longrightarrow M$$

Here, $T^*_{\leqslant\lambda} S^2 \subset T^*S^2$ comprises cotangent vectors of length $\leqslant \lambda$, where $\lambda > 0$ is a small parameter. With a suitable choice of $r(t)$ such that $r(t) = 0$ for $t \geqslant \lambda/2$, we obtain a model Dehn twist τ supported within this subspace. Subsequently, we define the Dehn twist τ_L as follows:

$$\tau_L(x) = \begin{cases} i\tau i^{-1}(x) & x \in \mathrm{im}(i), \\ x & \text{otherwise.} \end{cases}$$

It's worth noting that while the construction is not strictly unique, it is unique up to symplectic isotopy.

As we proceed, following [19], we will refer to a symplectomorphism as "fragile" if it is smoothly isotopic to the identity but not symplectically isotopic. Similarly, we will denote a diffeomorphism as "potentially fragile" if we can find a compatible symplectic form on the ambient manifold such that the upgraded symplectomorphism becomes fragile. Note that locally τ_{S^2} is fragile in T^*S^2, and τ_{S^2} is potentially fragile in a closed 4-manifold M.

Recalling an insight from [19], we can seamlessly integrate this local construction into any Dehn twist, leading to the following corollary:

Corollary 3.1. *For any Lagrangian sphere L residing within a closed symplectic four-manifold M, it turns out that the square τ_L^2 of the Dehn twist is potentially fragile.* □

Now, let's delve into an elementary construction directly based on the circle action σ employed in defining the Dehn twist.

Lemma 3.2. *Suppose that there exists a Hamiltonian circle action $\bar{\sigma}$ on $M \setminus L$ and a Weinstein neighborhood $i : T^*_{<\lambda} S^2 \to M$ of L that preserves equivariance with respect to σ and $\bar{\sigma}$. In this scenario, it follows that τ_L^2 is isotopic to the identity within $Symp(M)$.* □

3.1. **The Family of $M(r, 5)$: Two Extreme Cases.** $M(r, 5)$ emerges as the result of the diagonal action of $SO(3)$ on $(S^2)^5$, with a moment map defined as $\mu(x) = -(e_1 + \cdots + e_5)$. The topology could be any rational surface with Euler number at most 7. Two interesting cases of $M(r, 5)$ come into focus, offering extreme chambers as we vary the weights:

Example 3.3. *Consider $M = S^2 \times S^2$ equipped with a monotone symplectic form, and let $L = \{e_1 + e_2 = 0\}$ represent the antidiagonal. The diagonal $SO(3)$-action with the moment map $\mu(e) = -e_1 - e_2 \in \mathbb{R}^3$, and from the bending flow we know that $\bar{h}(e) = ||e_1 + e_2||$ is the moment map for a circle action $\bar{\sigma}$ on $M \setminus L$. This has the desired* *property with respect to any $SO(3)$-equivariant Lagrangian tubular neighborhood for L. Lemma 3.2 shows that τ_L^2 is isotopic to id, recovering Gromov's Theorem [6].*

Indeed τ_L is symplectically isotopic to the involution $(e_1, e_2) \mapsto (e_2, e_1)$. Notice that quotient this involution we get the Example 4.10.

Example 3.4. *Let's take $M \cong \mathbb{C}P^2\#4\overline{\mathbb{C}P^2}$, as established in Example 2.5 of [19], where M is known to be monotone.*

Define $L_i = \{e_i + e_{i+1} = 0\}$, and we observe that $M\setminus(L_i)$ carries a S^1-action. mirroring the construction in the previous example. Again, the bending flow gives a circle action by rotating e_1 around the axis formed by $e_1 + e_2$ while leaving $e_1 + e_2$, e_3, e_4, e_5 fixed. The relevant moment map is $h_1(e) = ||e_1 + e_2||$, and this is a circle action on $T^*L_1 \setminus L_1$ leaving L_1 fixed.

Much like the previous example, the squar square Dehn twist along each of the Lagrangian spheres L_i turns out to be isotopic to identity.

In conclusion, by varying this example and considering quintuples of vectors with different lengths (as detailed in [8, 5]), we can generate instances of Lagrangian spheres on $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$. In these cases, τ^2 is symplectically isotopic to the identity.

4. Projective twists in polygon spaces and related Gelfand-Celtin ssytems

Now we consider higher dimensional local symplectomrophisms that naturally generalize Dehn twists. This is a first attempt toward a special case of Conjecture 1.2:

Conjecture 4.1. *If $(M(r,n), \omega)$ admits a Hamiltonian torus action, then $Symp_h(M(r,n), \omega)$ is connected.*

4.1. Projective Twists.

In the context of a closed Riemannian manifold (L, g) with the property $H^1(L;\mathbb{R}) = 0$ and equipped with a periodic (co-)geodesic flow denoted as $\Phi_L^t : T^*L \to T^*L$, Seidel ([18]) introduces a class of symplectomorphisms in $\mathrm{Symp}_c(T^*L)$. We will provide an overview of the construction, using the notation from [14], for this class of symplectomorphisms, which we refer to as "twists." Specifically, when $L \cong S^{2n+1}$, this corresponds to the well-known symplectic "Dehn twist," and for cases where L falls within $\{\mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n\}$, this construction results in what we term a "projective twist."

For $L \cong S^n$ and given $\delta > 0$, we define an auxiliary function $r_\delta \in C^\infty([0,1], \mathbb{R})$ such that $0 < r_\delta(t) < \pi$ for all $t < \delta$, with the following behavior:

$$r_\delta(t) = \begin{cases} \frac{1}{2} - t & \text{if } t \ll \delta \\ 0 & \text{if } t \geqslant \delta \end{cases}$$

If L is a (real, complex, or quaternionic) projective space and $\delta > 0$, we let $r_\delta \in C^\infty([0,1], \mathbb{R})$ satisfy $0 < r_\delta(t) < 2\pi$ for all $t < \delta$, with the following behavior:

$$r_\delta(t) = \begin{cases} 1 - t & \text{if } t \ll \delta \\ 0 & \text{if } t \geqslant \delta \end{cases}$$

Here, $\|\cdot\|_L$ denotes the norm associated with the given Riemannian metric g. Consider the unit disc bundle $T^*_{<1}L$, where $T^*_{<s}L := \{v \in T^*L; \|v\|_L \leqslant s\}$, equipped with the standard symplectic form $\omega \in \Omega^2(T^*_{<1}L)$ and the contact form $\lambda \in \Omega^1(ST^*L)$, with ST^*L being the unit cotangent bundle.

The normalized co-geodesic flow Φ^t_L, which coincides with the Reeb flow for λ, satisfies $\Phi^1_L = \mathrm{Id}$ and can be extended to a Hamiltonian S^1-action denoted as σ_t on $D_1T^*L \setminus L$, with the moment map $\mu : T^*_{<1}L \to \mathbb{R}$ defined as $\mu(v) = \|v\|_L$.

Definition of Projective Twists

Parallel to the local twist along S^2, we define the model projective twist $\tau^{loc}_L \colon T^*_{<1}L \to T^*_{<1}L$ as follows:

Definition 4.2. *For L isomorphic to a projective space, we define:*

$$\tau^{loc}_L(p) = \begin{cases} \sigma_{r(\|p\|_L)}(p) & \text{if } p \notin L \\ p & \text{if } p \in L. \end{cases}$$

This summarizes the construction of these symplectomorphisms, which we term "twists," depending on the nature of the underlying manifold L.

Notice that the identification $S^2 \cong \mathbb{C}P^1$ induce identification $\tau^2_{S^2} \simeq \tau_{\mathbb{C}P^1}$.

Now suppose $L \subset M$ is a Lagrangian embedding of a Riemannian manifold L as above into a symplectic manifold (M, ω). By the Weinstein neighbourhood theorem, a neighbourhood of $L \subset M$ can be identified with a neighbourhood of $L \subset T^*L$, a disc bundle $D_{\leqslant s}T^*L$.

Definition 4.3. *Let $L \subset M$ be a exact Lagrangian submanifold embedded in M and τ^{loc}_L a model twist supported in the interior of $D^*_{\leqslant s}L$. Consider the symplectomorphism defined as*

$$\tau_L \cong \begin{cases} \iota \circ \tau^{loc}_L \circ \iota^{-1} & \text{on } \mathrm{Im}(\iota) \\ Id & \text{elsewhere} \end{cases}$$

In the case where L is a sphere, the map τ_L is the standard symplectic Dehn *twist. When L is a projective space, the resulting map is called* projective twist. *In this paper, the appellation* Dehn *is exclusively reserved for twists that are constructed from a Lagrangian sphere.*

Remark 4.4. *We refer to [18, Section 4.b] for the choices involved in this construction (in particular the auxiliary functions r_ϵ). Also, note that [18] proved that*

- *Given a symplectic manifold (M, ω), any Lagrangian $L \cong \mathbb{CP}^n \subset M$, the projective twist τ_L is smoothly trivial, i.e. it is isotopic to the identity in* $\mathrm{Diff}_c(M)$.
- *Let L be a simply connected projective manifold,then the local projective twist τ_L^{loc} is fragile, moreover, it has infinite order in $\pi_0(\mathrm{Symp}_c(T^*L))$.*

Proposition 4.5 (=Proposition 1.5)**.** *Let L be a projective Lagrangain. Suppose that there exists a Hamiltonian circle action $\bar{\sigma}$ on $M \setminus L$ and a Weinstein neighborhood $i : T^*_{<\lambda}L \to M$ of L that preserves equivariance with respect to σ and $\bar{\sigma}$. Then τ_L^2 is isotopic to the identity within Symp(M).* □

Proof. This is a straight forward argument by the constructions of projective twists.

Let's assume the Lagrangian L is projective and its complement has a circle action.

Now let's take the unit disk bundle $T^*_{\leq 1}L$. By the construction of projective twists τ_L, there is a family of smooth maps f_t such that $f_0 = id$ and $f_1 = \tau_L$. Now we will take the circle action of $M \setminus T^*_{\leq 1}L$, and use a radio Hamiltonian function to cut-off. The upshot is on the set $T^*_{=1}L$, the circle action is the one induced from $M \setminus T^*_{\leq 1}L$; while on each level $T^*_{a,a<1}L$, the cut-off untwists the action τ_L.

□

Example 4.6. *Recall Example 2.4, the symplectic manifold is $\mathbb{CP}^n$ with the Fubuny-Study form ω. Let $\mathbb{R}P^n$ be the moduli space of planer polygons. It is is a Lagrangian, and indeed the real part of $\mathbb{C}P^n$. The complement has a circle action fixing the $\mathbb{RP}^n$: by the Biran decomposition cf. [2], the complement of real $\mathbb{RP}^n \subset \mathbb{CP}^n$ is a disk bundle over the degree 4 hypersurface and the circle action is the diagonal action of the toric $\mathbb{CP}^n$. Then the square projective Dehn twist $\tau^2_{\mathbb{RP}^n}$ is isotopic to identity in Symp$(\mathbb{CP}^n, \omega)$.*

Example 4.7. *Recall Example 2.6 and 2.8, the symplectic manifold is a blowup of $\mathbb{CP}^n$ with the Fubini-Study form ω at linear subspaces.. The Lagrangian $\mathbb{R}P^N$ in $\mathbb{C}P^n$ is still there, by the bending flow in [11] or Theorem 2.2, there is a circle action in the complement. Another point of view is that such a blowup is given by equivariant blowup with sizes smaller than the disk fiber of the Biran decomposition. Hence the $\mathbb{RP}^n$ is not affected by the blowup.*

Example 4.8. *Recall Example 2.9, when $m = 2n + 3$, the symplectic manifolds can be realized as blowups of $\mathbb{CP}^n \times \mathbb{CP}^n$ with the Fubini-Study form ω at linear subspaces. The Lagrangian $\mathbb{C}P^n$ in $\mathbb{C}P^n \times \mathbb{C}P^n$ persists under the blowup and by or Theorem 2.2, there is a circle action in the complement. Hence the square projective twist is isotopic to identity in Symp$(M(r), \omega)$, by Proposition 1.4.*

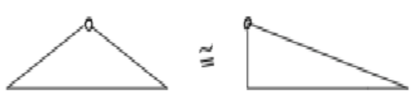
Another point of view is that such a blowup is given by equivariant blowup with sizes smaller than the disk fiber of the Biran decomposition. Hence the $\mathbb{RP}^n$ is not affected by the blowup.

Remark 4.9. *The above examples are not a complete listfor polygon spaces with projective Lagrangians. It does not cover the non generic weight vector cases, which could be toric degenerations(cf. [16]). It is possible to give a general pattern using Proposition 4.5 with a more careful analysis of the integrable systm*

Note that there is a volume formula (and computation of Chern classes) for the symplectic form constructed by the bending flow. However, an explicit descripsion of cohomology classes is still missing. We hope to explore both aspects in a future work **??**.

4.2. **Examples related to Gelfand-Celtin systems.** Note that $M(r, n)$ are rational varieties, which are naturally related to Grassmannians by the symplectic Gelfand-MacPherson correspondence in [8]. They are both important examples of Fano varieties and support integrable systems with generic fibers being torus and non-torus Lagrangian fibers. In spirit of Strominger-Yau-Zaslow and homological mirror symmetry, non-torus Lagrangian fibers is responsible for the incompleteness of the Givental-Hori-Vafa mirror cf. [**NNU10**]. in order to study closed mirror symmetry

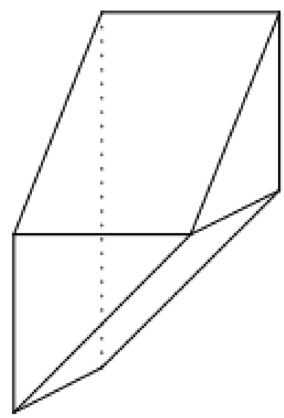
Example 4.10. *A Lagrangain $\mathbb{R}P^2 \subset \mathbb{C}P^2$.*



Notice that in this figure, the top circle is $\mathbb{R}P^2$, and the triangle is moment polytope (cf. [12]) of $\mathbb{C}P^2 \setminus \mathbb{R}P^2$. Note that the complement is a rational curve of degree 2 in $\mathbb{C}P^2$. There is a circle action in this complement by extending the circle actions on each edges. Notice that this is used in [3].

Note that this picture continue to hold for $\mathbb{CP}^n$, where $n \geq 3$. A related picture is the following: Let $Q_n := \{z_0^2 = z_1^2 + \cdots + z_n^2\} \subset \mathbb{C}P^n$. The real part of Q_n is a Lagrangian S^n.

Example 4.11. *A Lagrangian $S^3 \subset F(1, 2, 3)$.*



This is the Gelfand-Cetlin polytope of the full flag manifold $F(1,2,3)$. There is only one non-Delzant point, which is the vertex with 4 edges. At this point one has a Lagrangian S^3 (it comes from $S^3 \simeq SU(2)$.) In the complement there is a circle action, again by extending the circle actions on the edges. Notice that the Floer homology and disk potential of this fiber is computed in [16].

In general, the Gelfand-Celtin fiber may not be a projective Lagrangian. For example, the next flag manifold $Gr(2,4)$ has a Lagrangian $U(2)$ which does not all geodesics closed. However, there is a correspondence between the integrable system of Grassmannian and the bending flow system on the space of polygons. We end this section with the following question: when does a Gelfand-Celtin have special Lagrangian fiber being a projective Lagrangian?

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